

# ARRANGEMENTS OF IDEAL TYPE

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**ABSTRACT.** In 2006 Sommers and Tymoczko defined so called arrangements of ideal type  $\mathcal{A}_{\mathcal{I}}$  stemming from ideals  $\mathcal{I}$  in the set of positive roots of a reduced root system. They showed in a case by case argument that  $\mathcal{A}_{\mathcal{I}}$  is free if the root system is of classical type or  $G_2$  and conjectured that this is also the case for all types. This was established only very recently in a uniform manner by Abe, Barakat, Cuntz, Hoge and Terao. The set of non-zero exponents of the free arrangement  $\mathcal{A}_{\mathcal{I}}$  is given by the dual of the height partition of the roots in the complement of  $\mathcal{I}$  in the set of positive roots, generalizing the Shapiro-Steinberg-Kostant theorem which asserts that the dual of the height partition of the set of positive roots gives the exponents of the associated Weyl group.

Our first aim in this paper is to investigate a stronger freeness property of the  $\mathcal{A}_{\mathcal{I}}$ . We show that all  $\mathcal{A}_{\mathcal{I}}$  are inductively free, with the possible exception of some cases in type  $E_8$ .

In the same paper from 2006, Sommers and Tymoczko define a Poincaré polynomial  $\mathcal{I}(t)$  associated with each ideal  $\mathcal{I}$  which generalizes the Poincaré polynomial  $W(t)$  for the underlying Weyl group  $W$ . Solomon showed that  $W(t)$  satisfies a product decomposition depending on the exponents of  $W$  for any Coxeter group  $W$ . Sommers and Tymoczko showed in a case by case analysis in type  $A_n$ ,  $B_n$  and  $C_n$ , and some small rank exceptional types that a similar factorization property holds for the Poincaré polynomials  $\mathcal{I}(t)$  generalizing the formula of Solomon for  $W(t)$ . They conjectured that their multiplicative formula for  $\mathcal{I}(t)$  holds in all types. In our second aim to investigate this conjecture further, the same inductive tools we develop to obtain inductive freeness of the  $\mathcal{A}_{\mathcal{I}}$  are also employed. Here we also show that this conjecture holds inductively in almost all instances with only a small number of possible exceptions.

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## 1. INTRODUCTION

Much of the motivation for the study of arrangements of hyperplanes comes from Coxeter arrangements. They consist of the reflecting hyperplanes associated with the reflections

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of the underlying Coxeter group. While Coxeter arrangements are a well studied subject, subarrangements of the latter are considerably less well understood. In this paper we study certain arrangements which are associated with ideals in the set of positive roots of a reduced root system, so called *arrangements of ideal type*  $\mathcal{A}_{\mathcal{I}}$ , Definition 1.3, cf. [ST06, §11].

**1.1. Ideals in  $\Phi^+$ .** Let  $\Phi$  be an irreducible, reduced root system and let  $\Phi^+$  be the set of positive roots with respect to some set of simple roots  $\Pi$ . An (*upper*) *order ideal*, or simply *ideal* for short, of  $\Phi^+$ , is a subset  $\mathcal{I}$  of  $\Phi^+$  satisfying the following condition: if  $\alpha \in \mathcal{I}$  and  $\beta \in \Phi^+$  so that  $\alpha + \beta \in \Phi^+$ , then  $\alpha + \beta \in \mathcal{I}$ .

Recall the standard partial ordering  $\preceq$  on  $\Phi$ :  $\alpha \preceq \beta$  provided  $\beta - \alpha$  is a  $\mathbb{Z}_{\geq 0}$ -linear combination of positive roots, or  $\beta = \alpha$ . Then  $\mathcal{I}$  is an ideal in  $\Phi^+$  if and only if whenever  $\alpha \in \mathcal{I}$  and  $\beta \in \Phi^+$  so that  $\alpha \preceq \beta$ , then  $\beta \in \mathcal{I}$ .

Let  $\beta$  be in  $\Phi^+$ . Then  $\beta = \sum_{\alpha \in \Pi} c_{\alpha} \alpha$  for  $c_{\alpha} \in \mathbb{Z}_{\geq 0}$ . The *height* of  $\beta$  is defined to be  $\text{ht}(\beta) = \sum_{\alpha \in \Pi} c_{\alpha}$ .

Let  $\mathcal{I} \subseteq \Phi^+$  be an ideal and let

$$\mathcal{I}^c := \Phi^+ \setminus \mathcal{I}$$

be its complement in  $\Phi^+$ . Further, define  $\lambda_i := |\{\alpha \in \mathcal{I}^c \mid \text{ht}(\alpha) = i\}|$ . This gives the *height partition*  $\lambda_1 \geq \lambda_2 \geq \dots$  of  $\mathcal{I}^c$ . Let  $s = \lambda_1$ , the number of simple roots in  $\mathcal{I}^c$ .

**Definition 1.1** ([ST06, Def. 3.2]). The *ideal exponents*  $m_i^{\mathcal{I}}$  of an ideal  $\mathcal{I}$  in  $\Phi^+$  are the parts of the dual of the height partition of  $\mathcal{I}^c$ , i.e.  $m_i^{\mathcal{I}} := |\{\lambda_j \mid \lambda_j \geq s - i + 1\}|$ , so that

$$(1.2) \quad m_s^{\mathcal{I}} \geq \dots \geq m_1^{\mathcal{I}}.$$

Note that since  $\lambda_1 > \lambda_2$ , we have  $m_1^{\mathcal{I}} = 1$ , cf. [ST06, Prop. 3.1].

This terminology is motivated as follows. For  $\mathcal{I} = \emptyset$ , we have  $\mathcal{I}^c = \Phi^+$ . A famous theorem asserts that the dual of the height partition of  $\Phi^+$  gives the exponents of  $W$ . This connection was first discovered by Shapiro (unpublished) and rediscovered independently by Steinberg [Ste59, §9]. Kostant [Ko59] was the first to provide a uniform proof. Macdonald gave a proof using generating functions [Mac72].

**1.2. Arrangements of ideal type.** Following [ST06, §11], we associate with an ideal  $\mathcal{I}$  in  $\Phi^+$  the arrangement consisting of all hyperplanes with respect to the roots in  $\mathcal{I}^c$ . Let  $\mathcal{A}(\Phi)$  be the *Weyl arrangement* of  $\Phi$ , i.e.,  $\mathcal{A}(\Phi) = \{H_{\alpha} \mid \alpha \in \Phi^+\}$ , where  $H_{\alpha}$  is the hyperplane in the Euclidean space  $V = \mathbb{R} \otimes \mathbb{Z}\Phi$  orthogonal to the root  $\alpha$ .

**Definition 1.3** ([ST06, §11]). Let  $\mathcal{I} \subseteq \Phi^+$  be an ideal. The *arrangement of ideal type* associated with  $\mathcal{I}$  is the subarrangement  $\mathcal{A}_{\mathcal{I}}$  of  $\mathcal{A}(\Phi)$  defined by

$$\mathcal{A}_{\mathcal{I}} := \{H_{\alpha} \mid \alpha \in \mathcal{I}^c\}.$$

It was shown by Sommers and Tymoczko [ST06, Thm. 11.1] that in case the root system is classical or of type  $G_2$ , each  $\mathcal{A}_{\mathcal{I}}$  is free and the non-zero exponents are given by the ideal exponents of  $\mathcal{I}$ , cf. (1.2). The general case was settled only recently in a uniform manner for all types by Abe, Barakat, Cuntz, Hoge, and Terao in [ABC<sup>+</sup>16, Thm. 1.1].

**Theorem 1.4** ([ST06, Thm. 11.1], [ABC<sup>+</sup>16, Thm. 1.1]). *Let  $\Phi$  be a reduced root system with Weyl arrangement  $\mathcal{A} = \mathcal{A}(\Phi)$ . Then any subarrangement of  $\mathcal{A}$  of ideal type  $\mathcal{A}_{\mathcal{I}}$  is free with the non-zero exponents given by the ideal exponents  $m_i^{\mathcal{I}}$  of  $\mathcal{I}$ .*

The method of proof of [ABC<sup>+</sup>16, Thm. 1.1] entails a new general version of Terao's seminal addition deletion theorem (Theorem 2.3) allowing for an entire set of hyperplanes to be added at once to a given free arrangement while retaining freeness - given suitable circumstances - as opposed to adding just one hyperplane at a time. In particular, this method implies that there is a total order on the set of hyperplanes in  $\mathcal{A}_{\mathcal{I}}$  such that each term of the resulting chain of subarrangements of  $\mathcal{A}_{\mathcal{I}}$  is itself free. See [AT16] for an application of this method in the context of Shi arrangements.

The fact that each  $\mathcal{A}_{\mathcal{I}}$  is free with the non-zero exponents given by the ideal exponents further justifies the choice of terminology in Definition 1.1. The proof of [ABC<sup>+</sup>16, Thm. 1.1] is stunning, for not only does it generalize the aforementioned result by Shapiro-Steinberg-Kostant, it also gives a new uniform proof of the latter.

Thanks to independent fundamental work of Arnold and Saito, see [OT92, §6], the reflection arrangement  $\mathcal{A}(W)$  of any real reflection group  $W$  is free. In [BC12, Cor. 5.15], Barakat and Cuntz showed that in fact  $\mathcal{A}(W)$  is *inductively free*, see Definition 2.4. The most challenging case here is that of type  $E_8$ . In view of these results and considering the method of proof of Theorem 1.4 in [ABC<sup>+</sup>16, Thm. 1.1], it is natural to ask whether the free subarrangements  $\mathcal{A}_{\mathcal{I}}$  of  $\mathcal{A}(W)$  are also inductively free. Our first aim is to show that all  $\mathcal{A}_{\mathcal{I}}$  are indeed inductively free with the possible exception of some instances only in type  $E_8$ , see Theorem 1.15.

In recent work [Hul16], Hultman characterized all  $\mathcal{A}_{\mathcal{I}}$  that are *supersolvable*, see Definition 2.10. Since, supersolvability implies inductive freeness, see Theorem 2.15, the following result from [Hul16] readily provides a large collection of inductively free  $\mathcal{A}_{\mathcal{I}}$ .

**Theorem 1.5** ([Hul16, Thms. 6.6, 7.1]). *For  $\Phi$  of type  $A_n$ ,  $B_n$ ,  $C_n$ , or  $G_2$ , each  $\mathcal{A}_{\mathcal{I}}$  is supersolvable with the non-zero exponents given by the ideal exponents  $m_i^{\mathcal{I}}$  of  $\mathcal{I}$ .*

**Remark 1.6.** For  $\Phi$  of type  $A_n$ , every free subarrangement of  $\mathcal{A}$  is already supersolvable, [ER94, Thm. 3.3]. Thus in that case, Theorem 1.5 follows from Theorem 1.4.

As opposed to type  $A_n$ , a free subarrangement of the supersolvable arrangement of type  $B_n$  need not be supersolvable in general, e.g. the Weyl arrangement of type  $D_n$  is not supersolvable for  $n \geq 4$ , cf. Theorem 2.28(i). So the fact that all  $\mathcal{A}_{\mathcal{I}}$  in type  $B_n$  and type  $C_n$  are supersolvable is not a consequence of Theorem 1.4.

We give a short proof of Theorem 1.5 based on Theorem 1.12(i) in Section §3.

In contrast to the types covered in Theorem 1.5, there are always non-supersolvable arrangements of ideal type for the other Dynkin types, e.g., for  $\mathcal{I} = \emptyset$ , the Weyl arrangement  $\mathcal{A}_{\mathcal{I}} = \mathcal{A}(\Phi)$  itself is not supersolvable, see Theorem 2.28; see also [Hul16]. Nevertheless, in type  $D_n$  we obtain the following.

**Theorem 1.7.** *Let  $\Phi$  be of type  $D_n$  for  $n \geq 4$ . Then each  $\mathcal{A}_{\mathcal{I}}$  is inductively free with the non-zero exponents given by the ideal exponents  $m_i^{\mathcal{I}}$  of  $\mathcal{I}$ .*

From the last two results and Theorem 2.15, the following is immediate.

**Theorem 1.8.** *For  $\Phi$  of classical type, each  $\mathcal{A}_{\mathcal{I}}$  is inductively free with the non-zero exponents given by the ideal exponents  $m_i^{\mathcal{I}}$  of  $\mathcal{I}$ .*

For arbitrary  $\Phi$ , we obtain the following.

**Theorem 1.9.** *For  $\theta$  the highest root in  $\Phi^+$  and  $\mathcal{I} = \{\theta\}$ ,  $\mathcal{A}_{\mathcal{I}}$  is inductively free with the non-zero exponents given by the ideal exponents  $m_i^{\mathcal{I}}$  of  $\mathcal{I}$ .*

Next we describe an inductive tool that is a generalization of a criterion from [ST06, §7]. It is crucial in the proofs of both Theorems 1.5 and 1.7, as well as in all subsequent results. Using induction, it allows us to show that a large class of arrangements of ideal type is inductively free also in the exceptional types, see Theorems 1.13 and 1.14.

Let  $\Phi_0$  be a (standard) parabolic subsystem of  $\Phi$  (cf. §2.8) and let

$$\Phi_0^c := \Phi^+ \setminus \Phi_0^+,$$

the set of positive roots in the ambient root system which do not lie in the smaller one. The following is a generalization of an inductive criterion from [ST06, §7].

**Condition 1.10.** *Let  $\mathcal{I}$  be an ideal in  $\Phi^+$  and let  $\Phi_0$  be a maximal parabolic subsystem of  $\Phi$  such that  $\Phi_0^c \cap \mathcal{I}^c \neq \emptyset$ . Assume that firstly,  $\Phi_0^c \cap \mathcal{I}^c$  is linearly ordered with respect to  $\preceq$  so that there is a unique root of every occurring height in  $\Phi_0^c \cap \mathcal{I}^c$ , and secondly, for any  $\alpha \neq \beta$  in  $\Phi_0^c \cap \mathcal{I}^c$ , there is a  $\gamma \in \Phi_0^+$  so that  $\alpha, \beta$ , and  $\gamma$  are linearly dependent.*

**Definition 1.11.** Fix a standard parabolic subsystem  $\Phi_0$  of  $\Phi$ . For  $\mathcal{I}$  an ideal in  $\Phi^+$ ,

$$\mathcal{I}_0 := \mathcal{I} \cap \Phi_0^+$$

is an ideal in  $\Phi_0^+$ . Thus

$$\mathcal{A}_{\mathcal{I}_0} := \{H_\gamma \mid \gamma \in \mathcal{I}_0^c = \Phi_0^+ \setminus \mathcal{I}_0\}$$

is an arrangement of ideal type in  $\mathcal{A}(\Phi_0)$ , the Weyl arrangement of  $\Phi_0$ .

Obviously, since  $\mathcal{I}_0^c = \Phi_0^+ \setminus \mathcal{I}_0 = \mathcal{I}^c \cap \Phi_0^+ \subseteq \mathcal{I}^c$ , we may view  $\mathcal{A}_{\mathcal{I}_0}$  as a subarrangement of  $\mathcal{A}_{\mathcal{I}}$  rather than as a subarrangement of  $\mathcal{A}(\Phi_0)$ . Note however, as such,  $\mathcal{A}_{\mathcal{I}_0}$  is not of ideal type in  $\mathcal{A}$  in general, since  $\mathcal{I}_0$  need not be an ideal in  $\Phi^+$ .

Our next result shows that Condition 1.10 allows us to derive various stronger freeness properties of  $\mathcal{A}_{\mathcal{I}}$  from those of  $\mathcal{A}_{\mathcal{I}_0}$ . This is the principal inductive tool in our entire study.

For the notion of an *inductively factored* arrangement, see Definition 2.19. This also implies inductive freeness, see Proposition 2.21.

**Theorem 1.12.** *Let  $\mathcal{I}$  be an ideal in  $\Phi^+$  and let  $\Phi_0$  be a maximal parabolic subsystem of  $\Phi$  such that either  $\Phi_0^c \cap \mathcal{I}^c = \emptyset$  or else Condition 1.10 is satisfied. Then the following hold:*

- (i)  $\mathcal{A}_{\mathcal{I}_0}$  is supersolvable if and only if  $\mathcal{A}_{\mathcal{I}}$  is supersolvable;
- (ii)  $\mathcal{A}_{\mathcal{I}_0}$  is inductively free if and only if  $\mathcal{A}_{\mathcal{I}}$  is inductively free;
- (iii)  $\mathcal{A}_{\mathcal{I}_0}$  is inductively factored if and only if  $\mathcal{A}_{\mathcal{I}}$  is inductively factored.

*In particular, in each of the cases above we have*

$$\exp \mathcal{A}_{\mathcal{I}} = \{\exp \mathcal{A}_{\mathcal{I}_0}, |\mathcal{A}_{\mathcal{I}} \setminus \mathcal{A}_{\mathcal{I}_0}|\} = \{0^{n-k}, m_1^{\mathcal{I}}, \dots, m_s^{\mathcal{I}}\}.$$

The key observation for the proof of Theorem 1.12 stems from the fact that Condition 1.10 entails the presence of a modular element in  $L(\mathcal{A}_{\mathcal{I}})$  of rank  $r(\mathcal{A}_{\mathcal{I}}) - 1$  so that  $\mathcal{A}_{\mathcal{I}_0}$  (viewed as a subarrangement of  $\mathcal{A}_{\mathcal{I}}$ ) is realized as the localization of  $\mathcal{A}_{\mathcal{I}}$  with respect to this modular element, see Lemmas 3.1 and 3.4.

We emphasize that our proofs of Theorems 1.5 and 1.7 crucially depend on Theorem 1.12, and so do our next two results which are central concerning inductive freeness of the  $\mathcal{A}_{\mathcal{I}}$ .

**Theorem 1.13.** *Let  $\Phi$  be a reduced root system with Weyl arrangement  $\mathcal{A} = \mathcal{A}(\Phi)$ . Let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . Suppose that either  $\mathcal{A}_{\mathcal{I}}$  is reducible, or else  $\mathcal{A}_{\mathcal{I}}$  is irreducible and there is a maximal parabolic subsystem of  $\Phi$  such that Condition 1.10 is satisfied. Suppose that each arrangement of ideal type for every proper parabolic subsystem is inductively free. Then the arrangement of ideal type  $\mathcal{A}_{\mathcal{I}}$  is inductively free with the non-zero exponents given by the ideal exponents  $m_i^{\mathcal{I}}$  of  $\mathcal{I}$ .*

Theorem 1.8 settles the question of inductive freeness of all  $\mathcal{A}_{\mathcal{I}}$  for classical root systems. Our next results addresses the situation for the exceptional types.

**Theorem 1.14.** *Let  $\Phi$  be a root system of exceptional type with Weyl arrangement  $\mathcal{A} = \mathcal{A}(\Phi)$ . Let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . Suppose that  $\mathcal{I}$  satisfies one of the following conditions*

- (i)  $\mathcal{A}_{\mathcal{I}}$  is reducible;
- (ii)  $\mathcal{A}_{\mathcal{I}}$  is irreducible and there is a maximal parabolic subsystem of  $\Phi$  such that Condition 1.10 is satisfied; or
- (iii)  $\mathcal{I} = \{\theta\}$  or  $\mathcal{I} = \emptyset$ .

*Suppose that each arrangement of ideal type for every proper parabolic subsystem of exceptional type is inductively free. Then  $\mathcal{A}_{\mathcal{I}}$  is inductively free.*

Table 1 gives the number of all  $\mathcal{A}_{\mathcal{I}}$  which satisfy one of the conditions above.

Specifically, in Table 1 we present the number of all arrangements of ideal type for each exceptional type in the first row, see (2.29). In the second row, we list the number of all  $\mathcal{A}_{\mathcal{I}}$  which satisfy one of the conditions in Theorem 1.14 and thus are inductively free under the assumption that this is the case for every proper parabolic subsystem of exceptional type.

$\Phi$	$E_6$	$E_7$	$E_8$	$F_4$	$G_2$
all $\mathcal{A}_{\mathcal{I}}$	833	4160	25080	105	8
ind. free $\mathcal{A}_{\mathcal{I}}$	771	3433	18902	85	8

TABLE 1. Ind. free  $\mathcal{A}_{\mathcal{I}}$  for exceptional  $\Phi$  from Theorem 1.14

Thanks to Theorems 1.8 and 1.14, it is evident from Table 1 that with the possible exception of a relatively small number of cases in the exceptional types, all  $\mathcal{A}_{\mathcal{I}}$  are inductively free. T. Hoge was able to check on a computer that all of the possible exceptions in type  $F_4$  (20 cases),  $E_6$  (62 cases) and  $E_7$  (727 cases) are indeed inductively free. There are 6178 undecided cases in  $E_8$  at present. Therefore, using Theorems 1.8 and 1.14 along with these computational results, we can derive the following.

**Theorem 1.15.** *Let  $\Phi$  be a reduced root system with Weyl arrangement  $\mathcal{A} = \mathcal{A}(\Phi)$ . Let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . Then  $\mathcal{A}_{\mathcal{I}}$  is inductively free with the non-zero exponents given by the ideal exponents  $m_i^{\mathcal{I}}$  of  $\mathcal{I}$  with the possible exception when  $W$  is of type  $E_8$  and  $\mathcal{I}$  is one of 6178 ideals in  $\Phi^+$ .*

This leads to the following conjecture.

**Conjecture 1.16.** *Let  $\Phi$  be a reduced root system with Weyl arrangement  $\mathcal{A} = \mathcal{A}(\Phi)$ . Then any subarrangement of  $\mathcal{A}$  of ideal type  $\mathcal{A}_{\mathcal{I}}$  is inductively free with the non-zero exponents given by the ideal exponents  $m_i^{\mathcal{I}}$  of  $\mathcal{I}$ .*

**Remarks 1.17.** (i). We emphasize that our proofs of Theorems 1.5 – 1.15 do not depend on Theorem 1.4. Instead our arguments are based on extensions of ideas from the paper of Sommers and Tymoczko [ST06]. Condition 1.10 is clearly inspired by their work. Nevertheless, our approach is developed more within the framework of hyperplane arrangements and depends less on the root system combinatorics.

(ii). The freeness property asserted in Conjecture 1.16 is the strongest one we can hope to hold for *all*  $\mathcal{A}_{\mathcal{I}}$  for *all* types, as the Weyl arrangement  $\mathcal{A}(\Phi)$  itself is supersolvable (inductively factored) only if  $\Phi$  is of type  $A_n$ ,  $B_n$ ,  $C_n$ , or  $G_2$ , cf. Theorem 2.28.

(iii). Since inductively free arrangements are *divisionally free*, see [Abe16], it follows from Theorem 1.15 that with the possible exception of 6178 instances in type  $E_8$  all  $\mathcal{A}_{\mathcal{I}}$  are also divisionally free. Note, Conjecture 1.16 would settle affirmatively a conjecture by Abe that all arrangements of ideal type are divisionally free, [Abe16, Conj. 6.6].

(iv). The fact that T. Hoge was able to confirm by computer calculations that the remaining instances from Table 1 in types  $F_4$ ,  $E_6$  and  $E_7$  are indeed inductively free, suggests that the outstanding 6178 instances for  $E_8$  might also be in reach by suitable computational means. Although the number of undecided instances for  $E_8$  is small in relative terms (only 6178 out of 25080 cases), in view of the challenges the authors of [BC12] were faced with in connection with their computational proof of the inductive freeness of the full Weyl arrangement for  $E_8$ , this is likely going to be a formidable task.

Even if this finite number of unresolved instances can be confirmed computationally, it would be very desirable to have a uniform and conceptual proof of Conjecture 1.16. This would then entail a conceptual proof of the fact that the Weyl arrangement for  $E_8$  itself is inductively free.

**1.3. The Poincaré polynomial of a Coxeter group.** Let  $W$  be the Coxeter group associated with the root system  $\Phi$  and let  $\Phi^+$  be the system of positive roots with respect to a base  $\Pi$ . For  $w \in W$ , we define

$$(1.18) \quad N(w) := \{\alpha \in \Phi^+ \mid w\alpha \in -\Phi^+\}.$$

Then  $|N(w)| = \ell(w)$ , where  $\ell$  is the usual length function of  $W$  with respect to the fixed set of generators of  $W$  corresponding to  $\Pi$ .

Let  $t$  be an indeterminate. The *Poincaré polynomial*  $W(t)$  of the Coxeter group  $W$  is defined by

$$(1.19) \quad W(t) := \sum_{w \in W} t^{|N(w)|} = \sum_{w \in W} t^{\ell(w)}.$$

The following factorization of  $W(t)$  is due to Solomon [Sol66]:

$$(1.20) \quad W(t) = \prod_{i=1}^n (1 + t + \dots + t^{e_i}),$$

where  $\{e_1, \dots, e_n\}$  is the set of exponents of  $W$ . See also Macdonald [Mac72].

In geometric terms,  $W(t^2)$  is the Poincaré polynomial of the flag manifold  $G/B$ , where  $G$  is a semisimple Lie group with Weyl group  $W$  and  $B$  is a Borel subgroup of  $G$ . The formula (1.20) then gives a well-known factorization of this Poincaré polynomial.



**1.4. The Poincaré polynomial of an ideal.** Fix a subset  $R \subset \Phi^+$ . Following [ST06, §4], we say that a subset  $S \subseteq R$  is *R-closed* provided if  $\alpha, \beta \in S$  and  $\alpha + \beta \in R$ , then also  $\alpha + \beta \in S$ . For  $\mathcal{I}$  an ideal in  $\Phi^+$ , we say  $S \subset \mathcal{I}^c$  is of *Weyl type for  $\mathcal{I}$*  provided both  $S$  and its complement in  $\mathcal{I}^c$  are  $\mathcal{I}^c$ -closed. Let  $\mathcal{W}^{\mathcal{I}}$  denote the set of all subsets of  $\mathcal{I}^c$  which are of Weyl type for  $\mathcal{I}$ . These sets generalize the sets  $N(w)$  defined in (1.18) above. For, thanks to [ST06, Prop. 6.1], each such set  $S$  of Weyl type for  $\mathcal{I}$  is of the form  $S = N(w) \cap \mathcal{I}^c$  for some  $w \in W$ . So in particular, if  $\mathcal{I} = \emptyset$ , so that  $\mathcal{I}^c = \Phi^+$ , the sets of Weyl type for  $\mathcal{I} = \emptyset$  are precisely the sets  $N(w)$  in  $\Phi^+$ ; see also [Ko61]. Therefore, because of the analogy with (1.19) and following [ST06, §§4, 6], we call

$$(1.21) \quad \mathcal{I}(t) := \sum_{S \in \mathcal{W}^{\mathcal{I}}} t^{|S|}$$

the *Poincaré polynomial of  $\mathcal{I}$* . We can now formulate a further conjecture due to Sommers and Tymoczko which asserts that the analogue of Solomon's multiplicative formula (1.20) for  $W(t)$  holds for  $\mathcal{I}(t)$ .

**Conjecture 1.22** ([ST06, §4]). *Let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . Then*

$$(1.23) \quad \mathcal{I}(t) = \prod_{i=1}^k (1 + t + \dots + t^{m_i^{\mathcal{I}}}),$$

with  $m_i^{\mathcal{I}}$  the ideal exponents of  $\mathcal{I}$  introduced in Definition 1.1.

For  $\mathcal{I} = \emptyset$  the identity (1.23) specializes to Solomon's formula (1.20). In [ST06, Thm. 4.1], Sommers and Tymoczko gave a proof of the identity (1.23) based on case by case arguments in the following cases along with computer checks for  $F_4$  and  $E_6$ .

**Theorem 1.24** ([ST06, Thm. 4.1]). *Let  $\Phi$  be of type  $A_n$ ,  $B_n$ ,  $C_n$ , or  $G_2$  and let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . Then  $\mathcal{I}(t)$  satisfies (1.23).*

We present an alternative proof of Theorem 1.24 based on Theorems 1.5 and 5.2 and (5.7), see Remark 5.6(iii). Conjecture 1.22 is still open for the infinite family of type  $D_n$  for  $n \geq 4$ , as well as for types  $E_7$  and  $E_8$ .

In [ST06, Thm. 9.2], Sommers and Tymoczko gave a uniform proof of (1.23) for all types for  $\mathcal{I} = \{\theta\}$  using an alternate formula for  $W(t)$  due to Macdonald, [Mac72, Cor. 2.5].

**Theorem 1.25.** *Let  $\theta$  be the highest root of  $\Phi^+$  and let  $\mathcal{I} = \{\theta\}$ . Then  $\mathcal{I}(t)$  satisfies (1.23).*

Thanks to work of Tymoczko [Tym07], for  $\Phi$  of classical type,  $\mathcal{I}(t^2)$  is the Poincaré polynomial of the *regular nilpotent Hessenberg variety*  $\mathcal{H}_{\mathcal{I}}$  associated with  $\mathcal{I}$ . It follows from Theorem 1.24 that for  $\Phi$  of type  $A_n$ ,  $B_n$ , or  $C_n$ , the Poincaré polynomial of  $\mathcal{H}_{\mathcal{I}}$  admits a factorization as in (1.23), see [ST06, Thm. 10.2].

Our second aim is to address Conjecture 1.22. In this context Condition 1.10 is also a crucial inductive tool. Here are our main results in this direction.

**Theorem 1.26.** *Let  $\Phi$  be a reduced root system with Weyl arrangement  $\mathcal{A} = \mathcal{A}(\Phi)$ . Let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . Suppose that either  $\mathcal{A}_{\mathcal{I}}$  is reducible, or else  $\mathcal{A}_{\mathcal{I}}$  is irreducible and there is a maximal parabolic subsystem of  $\Phi$  such that Condition 1.10 is satisfied. Suppose that for every proper parabolic subsystem of  $\Phi$  the Poincaré polynomials of all ideals factor as in (1.23). Then  $\mathcal{I}(t)$  also factors as in (1.23), and so Conjecture 1.22 holds in these instances.*

Thanks to Corollary 6.3, Conjecture 1.22 holds for  $D_4$ . So in principle, we can argue by induction for type  $D_n$  using Theorem 1.26. Inductively Theorem 1.26 covers the bulk of all instances in type  $D_n$  in the following sense. It is straightforward to see that there are just  $2^{n-2}$  ideals  $\mathcal{I}$  in  $D_n$  for which Condition 1.10 is not satisfied with respect to  $\Phi_0$  being the standard subsystem of type  $D_{n-1}$ . In addition, in three of these  $2^{n-2}$  instances Conjecture 1.22 always holds, see Remark 6.4. This means, assuming the factorization result for  $D_{n-1}$ , it follows from Theorem 1.26 that it holds for all but  $2^{n-2} - 3$  instances in  $D_n$  as well. Nevertheless, the general case for type  $D_n$  is still unresolved.

The following settles Conjecture 1.22 for most instances in the exceptional types.

**Theorem 1.27.** *Let  $\Phi$  be an irreducible root system of exceptional type with Weyl arrangement  $\mathcal{A} = \mathcal{A}(\Phi)$ . Let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . Suppose that  $\mathcal{I}$  satisfies one of the following conditions*

- (i)  $\mathcal{A}_{\mathcal{I}}$  is reducible;
- (ii)  $\mathcal{A}_{\mathcal{I}}$  is irreducible and there is a maximal parabolic subsystem of  $\Phi$  such that Condition 1.10 is satisfied; or
- (iii)  $\mathcal{I} = \{\theta\}$  or  $\mathcal{I} = \emptyset$ .

*Suppose that for every proper parabolic subsystem of  $\Phi$  the Poincaré polynomials of all ideals factor as in (1.23). Then  $\mathcal{I}(t)$  also factors as in (1.23) and so Conjecture 1.22 holds in these instances.*

*Table 1 gives the number of all  $\mathcal{A}_{\mathcal{I}}$  which satisfy one of the conditions above.*

For  $\Phi$  of exceptional type, we see from the data in Table 1 that there is a large number of cases when it does follow from Theorem 1.27 that  $\mathcal{I}(t)$  factors as in (1.23). Sommers and Tymoczko have already checked computationally that Conjecture 1.22 holds in types  $F_4$  and  $E_6$ . It also holds for  $D_4$ , by Corollary 6.3. A. Schauenburg was able to confirm this also for root systems of type  $D_n$  for  $5 \leq n \leq 7$  and  $E_7$  by direct computation. Combining these computational results with Theorem 1.27, we obtain the following.

**Theorem 1.28.** *Let  $\Phi$  be an irreducible root system of exceptional type with Weyl arrangement  $\mathcal{A} = \mathcal{A}(\Phi)$ . Let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . Then Conjecture 1.22 holds for  $\mathcal{I}$  with the possible exception when  $W$  is of type  $E_8$  and  $\mathcal{I}$  is one of 6178 ideals.*

**Remarks 1.29.** (i). Our proofs of Theorems 1.26 and 1.27 utilize the observation from [ST06, §12] that the Poincaré polynomial  $\mathcal{I}(t)$  associated with  $\mathcal{A}_{\mathcal{I}}$  coincides with the rank generating function of the poset of regions of  $\mathcal{A}_{\mathcal{I}}$ , see (5.7). This allows us to study the former by means of the latter.

(ii). Thanks to work of Björner, Edelman, and Ziegler [BEZ90, Thm. 4.4], respectively Jambu and Paris [JP95, Prop. 3.4, Thm. 6.1], the rank generating function of the poset of regions of a real arrangement which is supersolvable, respectively inductively factored, admits a multiplicative decomposition which is equivalent to (1.23) for an arrangement of ideal type, according to (5.7), see Theorem 5.2.

Therefore, there is independent interest in the supersolvable and inductively factored instances among the  $\mathcal{A}_{\mathcal{I}}$ . The former have already been characterized by Hultman in [Hul16].

(iii). In [ST06, §12], Sommers and Tymoczko speculate about an equivalence of both of their theorems [ST06, Thms. 4.1, 11.1], cf. Theorems 1.4 and 1.24. The fact that all  $\mathcal{A}_{\mathcal{I}}$  are supersolvable in these instances implies both results, cf. Theorems 1.5 and 5.2. Indeed,



the similarity of the formulations of Theorems 1.13, 1.14, 1.26, and 1.27 confirm a deeper parallelism between both conjectures. This connection appears to stem from Condition 1.10 and the fact that this in turn entails the presence of a modular element in  $L(\mathcal{A}_{\mathcal{I}})$  of rank  $r(\mathcal{A}_{\mathcal{I}}) - 1$  (cf. Lemmas 3.1 and 3.4), which ultimately furnishes the induction argument, by means of Theorem 1.12(ii) and Lemma 5.4.

**1.5. Supersolvable and inductively factored arrangements of ideal type.** It follows from [Hul16] that there are always non-supersolvable arrangements of ideal type in all Dynkin types other than the ones covered in Theorem 1.5. Likewise for the notion of inductive factoredness. In view of Conjecture 1.22 it is rather natural to investigate inductive factoredness for the  $\mathcal{A}_{\mathcal{I}}$ , cf. Remark 1.29(ii). Theorem 1.12(iii) proves to be an equally useful inductive tools in this regard. We give an indication for  $\Phi$  of type  $D_n$  for  $n \geq 4$  and  $E_6$ ,  $E_7$ , and  $E_8$ , where the situation is already quite different to the one covered in Theorem 1.5, as demonstrated in our next result.

Let  $\theta$  be the highest root in  $\Phi$ . Then  $h = \text{ht}(\theta) + 1$  is the *Coxeter number* of  $W$ . For  $1 \leq t \leq h$ , let  $\mathcal{I}_t$  be the ideal consisting of all roots of height at least  $t$ , i.e.

$$\mathcal{I}_t := \{\alpha \in \Phi^+ \mid \text{ht}(\alpha) \geq t\}$$

In particular, we have  $\mathcal{I}_1 = \Phi^+$  and  $\mathcal{I}_h = \emptyset$ .

**Theorem 1.30.** *Let  $\Phi$  be of type  $D_n$  for  $n \geq 4$ ,  $E_6$ ,  $E_7$ , or  $E_8$ . Let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . Then the following hold:*

- (i) *if  $\mathcal{I} \subseteq \mathcal{I}_4$ , then  $\mathcal{A}_{\mathcal{I}}$  is not supersolvable;*
- (ii) *if  $\mathcal{I} \subseteq \mathcal{I}_5$ , then  $\mathcal{A}_{\mathcal{I}}$  is not inductively factored.*

**Theorem 1.31.** *Let  $\Phi$  be of type  $D_n$  for  $n \geq 4$ ,  $E_6$ ,  $E_7$ , or  $E_8$ . Let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . Then the following hold:*

- (i) *if  $\mathcal{I} \supseteq \mathcal{I}_3$ , then  $\mathcal{A}_{\mathcal{I}}$  is supersolvable;*
- (ii) *if  $\mathcal{I} \supseteq \mathcal{I}_4$ , then  $\mathcal{A}_{\mathcal{I}}$  is inductively factored.*

Thanks to Theorem 2.15 and Proposition 2.21, in all instances covered in Theorem 1.31,  $\mathcal{A}_{\mathcal{I}}$  is inductively free.

While in types  $A_n$ ,  $B_n$ ,  $C_n$  and  $G_2$ , the notions of supersolvability and inductive factoredness coincide for all  $\mathcal{A}_{\mathcal{I}}$ , according to Theorem 1.5, in contrast, in type  $D_n$  for  $n \geq 4$ ,  $E_6$ ,  $E_7$ , or  $E_8$ , the arrangement  $\mathcal{A}_{\mathcal{I}_4}$  is inductively factored but not supersolvable, thanks to Theorems 1.30(i) and 1.31(ii). Note that Theorems 1.30(i) and 1.31(i) also both follow readily from [Hul16].

The paper is organized as follows. In §§2.1 – 2.2 we recall some basic terminology and introduce further notation on hyperplane arrangements and record some basic facts on modular elements in the lattice of intersections of an arrangement. This is followed by brief sections on the fundamental notion on free and inductively free arrangements, including Terao's addition deletion Theorem 2.3. In §§2.5 and 2.6, the concepts of supersolvable, nice and inductively factored arrangements are recalled.

It is worth noting that all of these properties above are inherited by arbitrary localizations. In Lemmas 2.7 and 2.24 and Corollary 2.13, we show that if  $X$  in  $L(\mathcal{A})$  is modular of rank  $r(\mathcal{A}) - 1$  and the localization  $\mathcal{A}_X$  satisfies any of these properties, then so does  $\mathcal{A}$  itself.

The number of all ideals  $\mathcal{I}$  in  $\Phi^+$  was first obtained in a case by case analysis by Shi [Shi97]. An expression for this number in closed form was proved by Cellini and Papi [CP00, CP02], see §2.9. We also record closed formulas for the former sequences for the classical types depending on the rank in Table 4. They coincide with the famous Catalan sequences. Concerning the mathematical ubiquity of the latter, see [Sta99].

In the main section of the paper, §3, we prove in Lemma 3.1 that  $\mathcal{A}_{\mathcal{I}_0}$  is always a localization of  $\mathcal{A}_{\mathcal{I}}$  and in Lemma 3.4 that if  $\mathcal{I} \subseteq \Phi^+$  and  $\Phi_0$  satisfy Condition 1.10, then the center of this localization is a modular element of rank  $r(\mathcal{A}_{\mathcal{I}}) - 1$ . This and further results stemming from this fact are then used to prove Theorem 1.12. Subsequently Theorems 1.13 and 1.5 are then derived as consequences of Theorem 1.12. This is followed by a proof of Theorem 1.7. All of the above crucially depend on Condition 1.10 and Theorem 1.12.

Section 4 is devoted to the proof of Theorem 1.14. This crucially depends again on Theorems 1.5 and 1.12.

In §5.1, we recall the basics on the rank-generating function  $\zeta(P(\mathcal{A}, B), t)$  of the poset of regions  $P(\mathcal{A}, B)$  of a real arrangement  $\mathcal{A}$ . In Theorem 5.2 we recall theorems of Björner, Edelman and Ziegler [BEZ90, Thm. 4.4], respectively Jambu and Paris [JP95, Prop. 3.4, Thm. 6.1], asserting that for  $\mathcal{A}$  supersolvable, respectively inductively factored,  $\zeta(P(\mathcal{A}, B), t)$  satisfies a factorization analogous to (1.23). The proof of [BEZ90, Thm. 4.4] is used in Lemma 5.4 to show that  $\zeta(P(\mathcal{A}, B), t)$  factors with respect to a localization of  $\mathcal{A}$  at a modular element of rank  $r(\mathcal{A}) - 1$ . This in turn allows us to derive the desired factorization of  $\zeta(P(\mathcal{A}, B), t)$  despite the absence of supersolvability or inductive factoredness of the ambient arrangement, cf. Example 5.5. Lemma 5.4 quickly furnishes the proofs of Theorems 1.26 and 1.27.

Finally, in §6, after classifying all supersolvable and all inductively factored  $\mathcal{A}_{\mathcal{I}}$  in type  $D_4$  in Lemma 6.1, we complete the proofs of Theorems 1.30 and 1.31. The proof of Theorem 1.31 utilizes Condition 1.10 and Theorem 1.12 once again.

For general information about arrangements, Weyl groups and root systems, we refer the reader to [Bou68] and [OT92].

## 2. RECOLLECTIONS AND PRELIMINARIES

**2.1. Hyperplane arrangements.** Let  $\mathbb{K}$  be a field and let  $V = \mathbb{K}^\ell$  be an  $\ell$ -dimensional  $\mathbb{K}$ -vector space. A *hyperplane arrangement*  $\mathcal{A} = (\mathcal{A}, V)$  in  $V$  is a finite collection of hyperplanes in  $V$  each containing the origin of  $V$ . We also use the term  $\ell$ -arrangement for  $\mathcal{A}$ . We denote the empty arrangement in  $V$  by  $\emptyset_\ell$ .

The *lattice*  $L(\mathcal{A})$  of  $\mathcal{A}$  is the set of subspaces of  $V$  of the form  $H_1 \cap \dots \cap H_i$  where  $\{H_1, \dots, H_i\}$  is a subset of  $\mathcal{A}$ . For  $X \in L(\mathcal{A})$ , we have two associated arrangements, firstly  $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\} \subseteq \mathcal{A}$ , the *localization of  $\mathcal{A}$  at  $X$* , and secondly, the *restriction of  $\mathcal{A}$  to  $X$* ,  $(\mathcal{A}^X, X)$ , where  $\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}$ . Note that  $V$  belongs to  $L(\mathcal{A})$  as the intersection of the empty collection of hyperplanes and  $\mathcal{A}^V = \mathcal{A}$ . The lattice  $L(\mathcal{A})$  is a partially ordered set by reverse inclusion:  $X \leq Y$  provided  $Y \subseteq X$  for  $X, Y \in L(\mathcal{A})$ .

More generally, for  $U$  an arbitrary subspace of  $V$ , define  $\mathcal{A}_U := \{H \in \mathcal{A} \mid U \subseteq H\} \subseteq \mathcal{A}$ , the *localization of  $\mathcal{A}$  at  $U$* . Note that for  $X = \cap_{H \in \mathcal{A}_U} H$ , we have  $\mathcal{A}_X = \mathcal{A}_U$  and  $X$  belongs to the intersection lattice of  $\mathcal{A}$ .

For  $\mathcal{A} \neq \emptyset_\ell$ , let  $H_0 \in \mathcal{A}$ . Define  $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$ , and  $\mathcal{A}'' := \mathcal{A}^{H_0} = \{H_0 \cap H \mid H \in \mathcal{A}'\}$ , the restriction of  $\mathcal{A}$  to  $H_0$ . Then  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  is a *triple* of arrangements, [OT92, Def. 1.14].

Throughout, we only consider arrangements  $\mathcal{A}$  such that  $0 \in H$  for each  $H$  in  $\mathcal{A}$ . These are called *central*. In that case the *center*  $T(\mathcal{A}) := \bigcap_{H \in \mathcal{A}} H$  of  $\mathcal{A}$  is the unique maximal element in  $L(\mathcal{A})$  with respect to the partial order. A *rank* function on  $L(\mathcal{A})$  is given by  $r(X) := \text{codim}_V(X)$ . The *rank* of  $\mathcal{A}$  is defined as  $r(\mathcal{A}) := r(T(\mathcal{A}))$ .

The *product*  $\mathcal{A} = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$  of two arrangements  $(\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2)$  is defined by

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 := \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\},$$

see [OT92, Def. 2.13].

**2.2. Modular elements in  $L(\mathcal{A})$ .** We say that  $X \in L(\mathcal{A})$  is *modular* provided  $X + Y \in L(\mathcal{A})$  for every  $Y \in L(\mathcal{A})$ , [OT92, Cor. 2.26]. We require the following characterization of modular members of  $L(\mathcal{A})$  of rank  $r - 1$ , see the proof of [BEZ90, Thm. 4.3].

**Lemma 2.1.** *Let  $\mathcal{A}$  be an arrangement of rank  $r$ . Suppose that  $X \in L(\mathcal{A})$  is of rank  $r - 1$ . Then  $X$  is modular if and only if for any two distinct  $H_1, H_2 \in \mathcal{A} \setminus \mathcal{A}_X$  there is a  $H_3 \in \mathcal{A}_X$  so that  $r(H_1 \cap H_2 \cap H_3) = 2$ , i.e.  $H_1, H_2, H_3$  are linearly dependent.*

Next we record a special case of a general fact about modular elements in a geometric lattice, cf. [Aig79, Prop. 2.42] or [OT92, Lem. 2.27].

**Lemma 2.2.** *Let  $\mathcal{A}$  be an arrangement of rank  $r$ . Suppose that  $X \in L(\mathcal{A})$  is modular of rank  $r - 1$ . Then the map  $L(\mathcal{A}_X) \rightarrow L(\mathcal{A}^H)$  given by  $Y \mapsto Y \cap H$  is a lattice isomorphism for any  $H \in \mathcal{A} \setminus \mathcal{A}_X$ . In particular,  $\mathcal{A}_X \cong \mathcal{A}^H$ .*

**2.3. Free hyperplane arrangements.** Let  $S = S(V^*)$  be the symmetric algebra of the dual space  $V^*$  of  $V$ . Let  $\text{Der}(S)$  be the  $S$ -module of  $\mathbb{K}$ -derivations of  $S$ . Since  $S$  is graded,  $\text{Der}(S)$  is a graded  $S$ -module.

Let  $\mathcal{A}$  be an arrangement in  $V$ . Then for  $H \in \mathcal{A}$  we fix  $\alpha_H \in V^*$  with  $H = \ker \alpha_H$ . The *defining polynomial*  $Q(\mathcal{A})$  of  $\mathcal{A}$  is given by  $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S$ . The *module of  $\mathcal{A}$ -derivations* of  $\mathcal{A}$  is defined by

$$D(\mathcal{A}) := \{\theta \in \text{Der}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}.$$

We say that  $\mathcal{A}$  is *free* if  $D(\mathcal{A})$  is a free  $S$ -module, cf. [OT92, §4].

If  $\mathcal{A}$  is a free arrangement, then the  $S$ -module  $D(\mathcal{A})$  admits a basis of  $\ell$  homogeneous derivations, say  $\theta_1, \dots, \theta_\ell$ , [OT92, Prop. 4.18]. While the  $\theta_i$ 's are not unique, their polynomial degrees  $\text{pdeg } \theta_i$  are unique (up to ordering). This multiset is the set of *exponents* of the free arrangement  $\mathcal{A}$  and is denoted by  $\exp \mathcal{A}$ .

Terao's celebrated *addition deletion theorem* which we recall next plays a pivotal role in the study of free arrangements, [OT92, §4].

**Theorem 2.3** ([Ter80]). *Suppose that  $\mathcal{A}$  is non-empty. Let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be a triple of arrangements. Then any two of the following statements imply the third:*

- (i)  $\mathcal{A}$  is free with  $\exp \mathcal{A} = \{b_1, \dots, b_{\ell-1}, b_\ell\}$ ;
- (ii)  $\mathcal{A}'$  is free with  $\exp \mathcal{A}' = \{b_1, \dots, b_{\ell-1}, b_\ell - 1\}$ ;
- (iii)  $\mathcal{A}''$  is free with  $\exp \mathcal{A}'' = \{b_1, \dots, b_{\ell-1}\}$ .

There are various stronger notions of freeness which we discuss in the following subsections.

**2.4. Inductively free arrangements.** Theorem 2.3 motivates the notion of *inductively free* arrangements, see [Ter80] or [OT92, Def. 4.53].

**Definition 2.4.** The class  $\mathcal{IF}$  of *inductively free* arrangements is the smallest class of arrangements subject to

- (i)  $\emptyset_\ell$  belongs to  $\mathcal{IF}$ , for every  $\ell \geq 0$ ;
- (ii) if there exists a hyperplane  $H_0 \in \mathcal{A}$  such that both  $\mathcal{A}'$  and  $\mathcal{A}''$  belong to  $\mathcal{IF}$ , and  $\exp \mathcal{A}'' \subseteq \exp \mathcal{A}'$ , then  $\mathcal{A}$  also belongs to  $\mathcal{IF}$ .

**Remark 2.5.** It is possible to describe an inductively free arrangement  $\mathcal{A}$  by means of a so called *induction table*, cf. [OT92, §4.3, p. 119]. In this process we start with an inductively free arrangement and add hyperplanes successively ensuring that part (ii) of Definition 2.4 is satisfied. This process is referred to as *induction of hyperplanes*. This procedure amounts to choosing a total order on  $\mathcal{A}$ , say  $\mathcal{A} = \{H_1, \dots, H_m\}$ , so that each of the subarrangements  $\mathcal{A}_i := \{H_1, \dots, H_i\}$  and each of the restrictions  $\mathcal{A}_i^{H_i}$  is inductively free for  $i = 1, \dots, m$ . In the associated induction table we record in the  $i$ -th row the information of the  $i$ -th step of this process, by listing  $\exp \mathcal{A}'_i = \exp \mathcal{A}_{i-1}$ ,  $H_i$ , as well as  $\exp \mathcal{A}''_i = \exp \mathcal{A}_i^{H_i}$ , for  $i = 1, \dots, m$ . For instance, see [OT92, Tables 4.1, 4.2], or Table 2 below.

The class of free arrangements is closed with respect to taking localizations, cf. [OT92, Thm. 4.37]. This also holds for the class  $\mathcal{IF}$ , [HRS17, Thm. 1.1].

**Proposition 2.6.** *If  $\mathcal{A}$  is inductively free, then so is  $\mathcal{A}_U$  for every subspace  $U$  in  $V$ .*

For  $X$  in  $L(\mathcal{A})$  modular of rank  $r - 1$ , we get the following converse to Proposition 2.6, see also [Slo15, Lem. 6.7].

**Lemma 2.7.** *Let  $\mathcal{A}$  be an arrangement of rank  $r$ . Suppose that  $X \in L(\mathcal{A})$  is modular of rank  $r - 1$ . If  $\mathcal{A}_X$  is inductively free, then so is  $\mathcal{A}$ . In particular, if  $\exp \mathcal{A}_X = \{0, e_1, \dots, e_{\ell-1}\}$ , then  $\exp \mathcal{A} = \{e_1, \dots, e_{\ell-1}, e_\ell\}$ , where  $e_\ell := |\mathcal{A} \setminus \mathcal{A}_X|$ .*

*Proof.* We argue by means of an induction table for  $\mathcal{A}$  starting with the inductively free subarrangement  $\mathcal{A}_X$  and adding hyperplanes from  $\mathcal{A} \setminus \mathcal{A}_X = \{H_1, \dots, H_{e_\ell}\}$  (in any fixed order) successively, see Remark 2.5. Let  $\mathcal{A}_i := \mathcal{A}_X \cup \{H_1, \dots, H_i\}$  for  $i = 1, \dots, e_\ell$ . Then, since  $X$  is modular of rank  $r - 1$  also in  $L(\mathcal{A}_i)$ , for  $i = 1, \dots, e_\ell$  (cf. the second part of the proof of [BEZ90, Thm. 4.3]), it follows from Lemma 2.2 that  $\mathcal{A}_i^{H_i}$  is isomorphic to  $\mathcal{A}_X$  for each  $i = 1, \dots, e_\ell$ .

$\exp \mathcal{A}'_i$	$H_i$	$\exp \mathcal{A}''_i$
$e_1, \dots, e_{\ell-1}, 0$	$H_1$	$e_1, \dots, e_{\ell-1}$
$e_1, \dots, e_{\ell-1}, 1$	$H_2$	$e_1, \dots, e_{\ell-1}$
$\vdots$	$\vdots$	$\vdots$
$e_1, \dots, e_{\ell-1}, e_\ell - 1$	$H_{e_\ell}$	$e_1, \dots, e_{\ell-1}$
$e_1, \dots, e_{\ell-1}, e_\ell$		

TABLE 2. Induction table for  $\mathcal{A}$  starting at  $\mathcal{A}_X$

It thus follows from Table 2 and a repeated application of the addition part of Theorem 2.3 that  $\mathcal{A}$  is inductively free with  $\exp \mathcal{A} = \{e_1, \dots, e_{\ell-1}, e_\ell\}$ .  $\square$

**Remark 2.8.** The same argument as the one in the proof of Lemma 2.7 shows that if  $X \in L(\mathcal{A})$  is modular of rank  $r - 1$  and  $\mathcal{A}_X$  is free, then so is  $\mathcal{A}$ .

Free arrangements behave well with respect to the product construction for arrangements, [OT92, Prop. 4.28]. This property descends to the class  $\mathcal{IF}$ , [HR15, Prop. 2.10].

**Proposition 2.9.** *Let  $\mathcal{A}_1, \mathcal{A}_2$  be two arrangements. Then  $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$  is inductively free if and only if both  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are inductively free.*

**2.5. Supersolvable arrangements.** The following notion is due to Stanley [Sta72].

**Definition 2.10.** Let  $\mathcal{A}$  be a central arrangement of rank  $r$ . We say that  $\mathcal{A}$  is *supersolvable* provided there is a maximal chain

$$V = X_0 < X_1 < \dots < X_{r-1} < X_r = T(\mathcal{A})$$

of modular elements  $X_i$  in  $L(\mathcal{A})$ , cf. [OT92, Def. 2.32].

The class of supersolvable arrangements is closed under localization, [Sta72].

**Proposition 2.11.** *If  $\mathcal{A}$  is supersolvable, then so is  $\mathcal{A}_U$  for every subspace  $U$  in  $V$ .*

We require the following characterization of supersolvable arrangements due to Björner, Edelman and Ziegler, [BEZ90, Thm. 4.3].

**Theorem 2.12.** *Every arrangement of rank at most 2 is supersolvable. Let  $\mathcal{A}$  be an arrangement of rank  $r \geq 3$ . Then  $\mathcal{A}$  is supersolvable if and only if  $\mathcal{A}$  can be written as the proper disjoint union of two subarrangements  $\mathcal{A} = \mathcal{A}_0 \amalg \mathcal{A}_1$ , such that  $\mathcal{A}_0$  is supersolvable of rank  $r - 1$  and for any  $H_1 \neq H_2$  in  $\mathcal{A}_1$ , there is a  $H_3$  in  $\mathcal{A}_0$  so that  $H_1 \cap H_2 \subseteq H_3$ .*

Thanks to Lemma 2.1 and Theorem 2.12, we get a converse to Proposition 2.11 for  $X$  in  $L(\mathcal{A})$  modular of rank  $r - 1$ . (This is just the reverse implication in Theorem 2.12.)

**Corollary 2.13.** *Let  $\mathcal{A}$  be an arrangement of rank  $r$ . Suppose that  $X \in L(\mathcal{A})$  is modular of rank  $r - 1$ . If  $\mathcal{A}_X$  is supersolvable, then so is  $\mathcal{A}$ . In particular, if  $\exp \mathcal{A}_X = \{0, e_1, \dots, e_{\ell-1}\}$ , then  $\exp \mathcal{A} = \{e_1, \dots, e_{\ell-1}, e_\ell\}$ , where  $e_\ell := |\mathcal{A} \setminus \mathcal{A}_X|$ .*

Also supersolvable arrangements are compatible with the product construction for arrangements.

**Proposition 2.14** ([HR14, Prop. 2.5]). *Let  $\mathcal{A}_1, \mathcal{A}_2$  be two arrangements. Then  $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$  is supersolvable if and only if both  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are supersolvable.*

The connection of this notion with freeness is due to Jambu and Terao.

**Theorem 2.15** ([JT84, Thm. 4.2]). *A supersolvable arrangement is inductively free.*

**2.6. Nice and inductively factored arrangements.** The notion of a *nice* or *factored* arrangement goes back to Terao [Ter92]. It generalizes the concept of a supersolvable arrangement, see [OST84, Thm. 5.3] and [OT92, Prop. 2.67, Thm. 3.81]. Terao's main motivation was to give a general combinatorial framework to deduce tensor factorizations of the underlying Orlik-Solomon algebra, see also [OT92, §3.3]. We recall the relevant notions from [Ter92] (cf. [OT92, §2.3]):

**Definition 2.16.** Let  $\pi = (\pi_1, \dots, \pi_s)$  be a partition of  $\mathcal{A}$ .

- (a)  $\pi$  is called *independent*, provided for any choice  $H_i \in \pi_i$  for  $1 \leq i \leq s$ , the resulting  $s$  hyperplanes are linearly independent, i.e.  $r(H_1 \cap \dots \cap H_s) = s$ .
- (b) Let  $X \in L(\mathcal{A})$ . The *induced partition*  $\pi_X$  of  $\mathcal{A}_X$  is given by the non-empty blocks of the form  $\pi_i \cap \mathcal{A}_X$ .
- (c)  $\pi$  is *nice* for  $\mathcal{A}$  or a *factorization* of  $\mathcal{A}$  provided
  - (i)  $\pi$  is independent, and
  - (ii) for each  $X \in L(\mathcal{A}) \setminus \{V\}$ , the induced partition  $\pi_X$  admits a block which is a singleton.

If  $\mathcal{A}$  admits a factorization, then we also say that  $\mathcal{A}$  is *factored* or *nice*.

**Remark 2.17.** If  $\mathcal{A}$  is non-empty and  $\pi$  is a nice partition of  $\mathcal{A}$ , then the non-empty parts of the induced partition  $\pi_X$  form a nice partition of  $\mathcal{A}_X$  for each  $X \in L(\mathcal{A}) \setminus \{V\}$ ; cf. the proof of [Ter92, Cor. 2.11].

Following Jambu and Paris [JP95], we introduce further notation. Suppose  $\mathcal{A}$  is not empty. Let  $\pi = (\pi_1, \dots, \pi_s)$  be a partition of  $\mathcal{A}$ . Let  $H_0 \in \pi_1$  and let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be the triple associated with  $H_0$ . Then  $\pi$  induces a partition  $\pi'$  of  $\mathcal{A}'$ , i.e. the non-empty subsets  $\pi_i \cap \mathcal{A}'$ . Note that since  $H_0 \in \pi_1$ , we have  $\pi_i \cap \mathcal{A}' = \pi_i$  for  $i = 2, \dots, s$ . Also, associated with  $\pi$  and  $H_0$ , we define the *restriction map*

$$\varrho := \varrho_{\pi, H_0} : \mathcal{A} \setminus \pi_1 \rightarrow \mathcal{A}'' \text{ given by } H \mapsto H \cap H_0$$

and set

$$\pi''_i := \varrho(\pi_i) = \{H \cap H_0 \mid H \in \pi_i\} \text{ for } 2 \leq i \leq s.$$

In general,  $\varrho$  need not be surjective nor injective. However, since we are only concerned with cases when  $\pi'' = (\pi''_2, \dots, \pi''_s)$  is a partition of  $\mathcal{A}''$ ,  $\varrho$  has to be onto and  $\varrho(\pi_i) \cap \varrho(\pi_j) = \emptyset$  for  $i \neq j$ .

The following gives an analogue of Teramo's addition deletion Theorem 2.3 for free arrangements for the class of nice arrangements.

**Theorem 2.18** ([HR16a, Thm. 3.5]). *Suppose that  $\mathcal{A} \neq \emptyset_\ell$ . Let  $\pi = (\pi_1, \dots, \pi_s)$  be a partition of  $\mathcal{A}$ . Let  $H_0 \in \pi_1$  and let  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  be the triple associated with  $H_0$ . Then any two of the following statements imply the third:*

- (i)  $\pi$  is nice for  $\mathcal{A}$ ;
- (ii)  $\pi'$  is nice for  $\mathcal{A}'$ ;
- (iii)  $\varrho : \mathcal{A} \setminus \pi_1 \rightarrow \mathcal{A}''$  is bijective and  $\pi''$  is nice for  $\mathcal{A}''$ .

The bijectivity condition on  $\varrho$  in the theorem is necessary, cf. [HR16a, Ex. 3.3]. Theorem 2.18 motivates the following stronger notion of factorization, cf. [JP95], [HR16a, Def. 3.8].

**Definition 2.19.** The class  $\mathcal{IFAC}$  of *inductively factored* arrangements is the smallest class of pairs  $(\mathcal{A}, \pi)$  of arrangements  $\mathcal{A}$  together with a partition  $\pi$  subject to

- (i)  $(\emptyset_\ell, ())$  belongs to  $\mathcal{IFAC}$  for each  $\ell \geq 0$ ;
- (ii) if there exists a partition  $\pi$  of  $\mathcal{A}$  and a hyperplane  $H_0 \in \pi_1$  such that for the triple  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  associated with  $H_0$  the restriction map  $\varrho = \varrho_{\pi, H_0} : \mathcal{A} \setminus \pi_1 \rightarrow \mathcal{A}''$  is bijective and for the induced partitions  $\pi'$  of  $\mathcal{A}'$  and  $\pi''$  of  $\mathcal{A}''$  both  $(\mathcal{A}', \pi')$  and  $(\mathcal{A}'', \pi'')$  belong to  $\mathcal{IFAC}$ , then  $(\mathcal{A}, \pi)$  also belongs to  $\mathcal{IFAC}$ .



If  $(\mathcal{A}, \pi)$  is in  $\mathcal{IFAC}$ , then we say that  $\mathcal{A}$  is *inductively factored with respect to  $\pi$* , or else that  $\pi$  is an *inductive factorization* of  $\mathcal{A}$ . Sometimes, we simply say  $\mathcal{A}$  is *inductively factored* without reference to a specific inductive factorization of  $\mathcal{A}$ .

The connection with the previous notions is as follows, [HR16a, Prop. 3.11].

**Proposition 2.20.** *If  $\mathcal{A}$  is supersolvable, then  $\mathcal{A}$  is inductively factored.*

**Proposition 2.21** ([JP95, Prop. 2.2], [HR16a, Prop. 3.14]). *Let  $\pi = (\pi_1, \dots, \pi_r)$  be an inductive factorization of  $\mathcal{A}$ . Then  $\mathcal{A}$  is inductively free with  $\exp \mathcal{A} = \{0^{\ell-r}, |\pi_1|, \dots, |\pi_r|\}$ .*

**Remark 2.22.** In analogy to  $\mathcal{IF}$ , for members in  $\mathcal{IFAC}$  one can present a so called induction table of factorizations, cf. [HR16a, Rem. 3.16].

The proof of Proposition 2.21 shows that if  $\pi$  is an inductive factorization of  $\mathcal{A}$  and  $H_0 \in \mathcal{A}$  is distinguished with respect to  $\pi$ , then the triple  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  with respect to  $H_0$  is a triple of inductively free arrangements. Thus an induction table of  $\mathcal{A}$  can be constructed, compatible with suitable inductive factorizations of the subarrangements  $\mathcal{A}_i$ .

Let  $\mathcal{A} = \{H_1, \dots, H_m\}$  be a choice of a total order on  $\mathcal{A}$ . Then, starting with the empty partition for  $\emptyset_\ell$ , we can attempt to build inductive factorizations  $\pi_i$  of  $\mathcal{A}_i$  consecutively, resulting in an inductive factorization  $\pi = \pi_m$  of  $\mathcal{A} = \mathcal{A}_m$ . This is achieved by invoking Theorem 2.18 repeatedly in order to derive that each  $\pi_i$  is an inductive factorization of  $\mathcal{A}_i$ .

We then add the inductive factorizations  $\pi_i$  of  $\mathcal{A}_i$  as additional data into an induction table for  $\mathcal{A}$ . The data in such an extended induction table together with the addition part of Theorem 2.18 then proves that  $\mathcal{A}$  is inductively factored. We refer to this technique as *induction of factorizations* and the corresponding table as an *induction table of factorizations* for  $\mathcal{A}$ . See for instance [HR16a, §3.3], or Table 3.

By Remark 2.17, the class of nice arrangements is closed with respect to taking localizations. This property restricts to the class  $\mathcal{IFAC}$ , [MR17, Thm. 1.1].

**Proposition 2.23.** *If  $\mathcal{A}$  is inductively factored, then so is  $\mathcal{A}_U$  for every subspace  $U$  in  $V$ .*

The following gives a converse to Proposition 2.23 for  $X$  in  $L(\mathcal{A})$  modular of rank  $r - 1$ .

**Lemma 2.24.** *Let  $\mathcal{A}$  be an arrangement of rank  $r$ . Suppose that  $X \in L(\mathcal{A})$  is modular of rank  $r - 1$ . If  $\mathcal{A}_X$  is inductively factored, then so is  $\mathcal{A}$ . In particular, if  $\exp \mathcal{A}_X = \{0, e_1, \dots, e_{\ell-1}\}$ , then  $\exp \mathcal{A} = \{e_1, \dots, e_{\ell-1}, e_\ell\}$ , where  $e_\ell := |\mathcal{A} \setminus \mathcal{A}_X|$ .*

*Proof.* Let  $\pi_X$  be an inductive factorization of  $\mathcal{A}_X$  and let  $\exp \mathcal{A}_X = \{0, e_1, \dots, e_{\ell-1}\}$ . Let  $\pi^X := \mathcal{A} \setminus \mathcal{A}_X = \{H_1, \dots, H_{e_\ell}\}$  (given in any fixed order). Then  $\pi := (\pi_X, \pi^X)$  is a partition of  $\mathcal{A}$ . We show that  $\pi$  is an inductive factorization of  $\mathcal{A}$  by means of the following induction table of factorizations starting with the inductive factorization  $\pi_X$  of  $\mathcal{A}_X$ , see Remark 2.22.

It follows from Lemma 2.2 applied to the consecutive triples in this induction table that each restriction in Table 3 is isomorphic to  $\mathcal{A}_X$  and the induced partition  $\pi_i'' = \overline{\pi_X}$  is in bijection with  $\pi_X$ . Thus each restriction along with the induced partition in this table is isomorphic to the pair  $(\mathcal{A}_X, \pi_X)$ , and thus is inductively factored. Consequently, it follows from Table 3, Theorem 2.18 and Remark 2.22 that  $\pi = (\pi_X, \pi^X)$  is an inductive factorization of  $\mathcal{A}$ . Observe, the notation here is consistent with the one introduced in Definition 2.16(b), for the partition of  $\mathcal{A}_X$  induced from  $\pi$  is just  $\pi_X$ , the one we started with.  $\square$

$\pi'_i$	$\exp \mathcal{A}'_i$	$H_i$	$\pi''_i$	$\exp \mathcal{A}''_i$
$\pi_X$	$e_1, \dots, e_{\ell-1}, 0$	$H_1$	$\overline{\pi_X}$	$e_1, \dots, e_{\ell-1}$
$\pi_X, \{H_1\}$	$e_1, \dots, e_{\ell-1}, 1$	$H_2$	$\overline{\pi_X}$	$e_1, \dots, e_{\ell-1}$
$\pi_X, \{H_1, H_2\}$	$e_1, \dots, e_{\ell-1}, 2$	$H_3$	$\overline{\pi_X}$	$e_1, \dots, e_{\ell-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\pi_X, \{H_1, H_2, \dots, H_{e_\ell-1}\}$	$e_1, \dots, e_{\ell-1}, e_\ell - 1$	$H_{e_\ell}$	$\overline{\pi_X}$	$e_1, \dots, e_{\ell-1}$
$\pi = (\pi_X, \pi^X)$	$e_1, \dots, e_{\ell-1}, e_\ell$			

TABLE 3. Induction table of factorizations for  $\mathcal{A}$  starting at  $\mathcal{A}_X$

**Remark 2.25.** The same argument in the proof of Lemma 2.24 shows that if  $X \in L(\mathcal{A})$  is modular of rank  $r - 1$  and  $\mathcal{A}_X$  is nice, then so is  $\mathcal{A}$  (albeit without any reference to exponents, as  $\mathcal{A}_X$  might not be free).

As for the previous stronger freeness properties, inductively factored arrangements are compatible with the product construction for arrangements.

**Proposition 2.26** ([HR16a, Prop. 3.30]). *Let  $\mathcal{A}_1, \mathcal{A}_2$  be two arrangements. Then  $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$  is inductively factored if and only if both  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are inductively factored.*

**Remark 2.27.** Observe that the implications in Lemmas 2.7, 2.24 and Corollary 2.13 fail without the modularity requirement on  $X$ . For let  $\mathcal{A}$  be a non-free 3-arrangement and let  $X \in L(\mathcal{A})$  be of rank 2. Then being supersolvable,  $\mathcal{A}_X$  satisfies each of the stronger freeness properties, but  $\mathcal{A}$  does not.

**2.7. Supersolvable Weyl arrangements.** In [BC12, Cor. 5.15], Barakat and Cuntz showed that every Weyl arrangement is inductively free. In [HR14, Thm. 1.2] and [HR16b, Cor. 1.4], all supersolvable, respectively inductively factored, reflection arrangements were classified. It is going to be useful to know all instances when a Weyl arrangement itself satisfies any of these stronger freeness properties. It turns out that for a Weyl arrangement  $\mathcal{A} = \mathcal{A}(\Phi)$ , supersolvability, niceness and inductive factoredness all coincide.

**Theorem 2.28** ([HR14, Thm. 1.2], [HR16b, Thm. 1.3, Cor. 1.4]). *Let  $\Phi$  be an irreducible reduced root system with Weyl arrangement  $\mathcal{A} = \mathcal{A}(\Phi)$ . Then*

- (i)  $\mathcal{A}$  is supersolvable if and only if  $\Phi$  is of type  $A_n, B_n, C_n$ , or  $G_2$ ;
- (ii)  $\mathcal{A}$  is inductively factored if and only if  $\mathcal{A}$  is supersolvable;
- (iii)  $\mathcal{A}$  is nice if and only if  $\mathcal{A}$  is inductively factored.

**2.8. Parabolic subsystems and parabolic subgroups.** Let  $\Phi$  be a reduced root system of rank  $n$  with Weyl group  $W$  and reflection arrangement  $\mathcal{A} = \mathcal{A}(\Phi) = \mathcal{A}(W)$ .

For  $w \in W$ , write  $\text{Fix}(w) := \{v \in V \mid wv = v\}$  for the fixed point subspace of  $w$ . For  $U \subseteq V$  a subspace, we define the *parabolic subgroup*  $W_U$  of  $W$  by  $W_U := \{w \in W \mid U \subseteq \text{Fix}(w)\}$ . By Steinberg's Theorem [Ste60, Thm. 1.5], for  $U \subseteq V$  a subspace, the parabolic subgroup  $W_U$  is itself a Coxeter group, generated by the reflections in  $W$  that are contained in  $W_U$ .

Let  $X = \cap_{H \in \mathcal{A}_U} H$ . Then  $\mathcal{A}_X = \mathcal{A}_U$  and  $X \in L(\mathcal{A})$ . Thus,  $\mathcal{A}(W_U) = \mathcal{A}_U = \mathcal{A}_X = \mathcal{A}(W_X)$ , by [OT92, Thm. 6.27, Cor. 6.28].

Let  $\Phi^+$  be the set of positive roots with respect to some set of simple roots  $\Pi$  of  $\Phi$ . For  $\Pi_0$  a proper subset of  $\Pi$ , the (*standard parabolic*) *subsystem* of  $\Phi$  generated by  $\Pi_0$  is  $\Phi_0 := \mathbb{Z}\Pi_0 \cap \Phi$ , cf. [Bou68, Ch. VI §1.7]. Define  $\Phi_0^+ := \Phi_0 \cap \Phi^+$ , the set of positive roots of  $\Phi_0$  with respect to  $\Pi_0$ . If the rank of  $\Phi_0$  is  $n - 1$ , then  $\Phi_0$  is said to be *maximal*.

Set  $X_0 := \cap_{\gamma \in \Phi_0^+} H_\gamma$ . Then  $\mathcal{A}(\Phi)_{X_0} = \mathcal{A}(\Phi_0)$ . Therefore, the reflection arrangement  $\mathcal{A}(W_{X_0})$  of the parabolic subgroup  $W_{X_0}$  is just  $\mathcal{A}(\Phi_0)$ , i.e.  $\Phi_0$  is the root system of  $W_{X_0}$ .

**2.9. On the number of ideals.** Suppose that  $\Phi$  is irreducible with Weyl group  $W$ . Let  $\theta$  be the highest root in  $\Phi$ . A closed formula for the number of all ideals  $\mathcal{I}$  in  $\Phi^+$  was given by Cellini and Papi [CP00, CP02]:

$$(2.29) \quad \frac{1}{|W|} \prod_{i=1}^n (h + e_i + 1),$$

where  $h = \text{ht}(\theta) + 1$  is the Coxeter number of  $\Phi$ , and  $e_1, \dots, e_n$  are the exponents of  $W$ .

Following [Som05], we call an ideal  $\mathcal{I}$  in  $\Phi^+$  *strictly positive* provided it satisfies  $\mathcal{I} \cap \Pi = \emptyset$ , i.e., provided it does not contain a simple root. Note that this includes the empty ideal. Sommers proved a closed formula for the number of strictly positive ideals in [Som05]:

$$(2.30) \quad \frac{1}{|W|} \prod_{i=1}^n (h + e_i - 1),$$

where the notation is as in (2.29).

It is also useful to have closed expressions in terms of the rank for the numbers above for the classical types. These along with the same numbers for the exceptional types belong to the famous Catalan sequences. For further information on the combinatorial aspects and ubiquity of the latter, see for instance [Sta99].

$\Phi$	$A_n$	$B_n$	$C_n$	$D_n$
all $\mathcal{I}$	$\frac{1}{n+2} \binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\binom{2n}{n}$	$\binom{2n-1}{n} + \binom{2n-2}{n}$
strictly positive $\mathcal{I}$	$\frac{1}{n+1} \binom{2n}{n}$	$\binom{2n-1}{n-1}$	$\binom{2n-1}{n-1}$	$\binom{2n-2}{n} + \binom{2n-3}{n}$

TABLE 4. The number of all  $\mathcal{I}$  and strictly positive  $\mathcal{I}$  for classical  $\Phi$

In Table 5 we present the number of ideals  $\mathcal{I}$  that lie in  $\mathcal{I}_t$  for  $t \geq 1$  for the irreducible root systems of exceptional type of rank at least 4. Thus the first two rows give the number of ideals in  $\Phi^+$ , respectively strictly positive ideals in  $\Phi^+$ , according to (2.29) and (2.30).

The last row in Table 5 labeled with  $*$  gives the number of ideals  $\mathcal{I}$  that are not covered by the inductive argument of Theorems 1.14 and 1.27, see §4.3 and §4.11, and §5.2. The significance of the last entry in each column above the row labeled with  $*$  is explained in §4.3.

$\Phi$	$F_4$	$E_6$	$E_7$	$E_8$
$\mathcal{I}$	105	833	4160	25080
$\mathcal{I} \subseteq \mathcal{I}_2$	66	418	2431	17342
$\mathcal{I} \subseteq \mathcal{I}_3$	48	254	1660	13395
$\mathcal{I} \subseteq \mathcal{I}_4$	36	150	1162	10714
$\mathcal{I} \subseteq \mathcal{I}_5$	22	62	726	8330
$\mathcal{I} \subseteq \mathcal{I}_6$			403	6623
$\mathcal{I} \subseteq \mathcal{I}_7$				4500
$*$	20	62	727	6178

TABLE 5. The number of ideals  $\mathcal{I}$  in  $\mathcal{I}_t$  for  $t \geq 1$

### 3. ARRANGEMENTS OF IDEAL TYPE

We maintain the notation and setup from the Introduction and §2. In particular, let  $\Phi$  be a reduced root system in the real  $n$ -space  $V$  with a fixed set of simple roots  $\Pi$ , so that  $|\Pi| = n$ , and corresponding set of positive roots  $\Phi^+$ . Throughout, let  $\mathcal{A} = \mathcal{A}(\Phi)$  be the Weyl arrangement of  $\Phi$ . Let  $\Phi_0$  be a proper parabolic subsystem of  $\Phi$ , let  $\mathcal{I}$  be an ideal in  $\Phi^+$  and let  $\mathcal{I}_0 = \mathcal{I} \cap \Phi_0^+$ . We start with an elementary but crucial observation.

**Lemma 3.1.** *Let  $\Phi$ ,  $\mathcal{I}$  and  $\Phi_0$  be as above. Then, viewing  $\mathcal{A}_{\mathcal{I}_0}$  as a subarrangement of  $\mathcal{A}_{\mathcal{I}}$ , we have  $\mathcal{A}_{\mathcal{I}_0} = (\mathcal{A}_{\mathcal{I}})_{X_0}$ , where  $X_0 := \cap_{\gamma \in \Phi_0^+} H_\gamma$ .*

*Proof.* As a subarrangement of  $\mathcal{A} = \mathcal{A}(\Phi)$ ,  $\mathcal{A}(\Phi_0)$  coincides with  $\mathcal{A}_{X_0}$ , cf. §2.8. Therefore, viewing  $\mathcal{A}_{\mathcal{I}_0}$  as a subarrangement of  $\mathcal{A}_{\mathcal{I}}$ , we have

$$\mathcal{A}_{\mathcal{I}_0} = \{H_\gamma \mid \gamma \in \mathcal{I}_0^c = \mathcal{I}^c \cap \Phi_0^+\} = \mathcal{A}_{\mathcal{I}} \cap \mathcal{A}_{X_0} = (\mathcal{A}_{\mathcal{I}})_{X_0},$$

as desired. Note that  $X_0$  need not belong to  $L(\mathcal{A}_{\mathcal{I}})$  in general; e.g. see Example 3.3 below.  $\square$

The following is immediate from Propositions 2.6, 2.11, 2.23 and Lemma 3.1.

**Corollary 3.2.** *Let  $\Phi$ ,  $\mathcal{I}$  and  $\Phi_0$  be as above.*

- (i) *If  $\mathcal{A}_{\mathcal{I}}$  is supersolvable, then so is  $\mathcal{A}_{\mathcal{I}_0}$ .*
- (ii) *If  $\mathcal{A}_{\mathcal{I}}$  is inductively free, then so is  $\mathcal{A}_{\mathcal{I}_0}$ .*
- (iii) *If  $\mathcal{A}_{\mathcal{I}}$  is inductively factored, then so is  $\mathcal{A}_{\mathcal{I}_0}$ .*

Before discussing consequences of Condition 1.10, we illustrate two instances in an easy example when this condition holds, respectively fails. Note that in this setting  $\Phi_0$  is assumed to be a maximal parabolic subsystem of  $\Phi$ .

**Example 3.3.** Let  $\Phi$  be of type  $A_3$  with simple roots  $\Pi = \{\alpha, \beta, \gamma\}$ . Let  $\mathcal{I}$  be the ideal in  $\Phi^+$  generated by  $\beta$ . Then  $\mathcal{I}^c = \{\alpha, \gamma\}$ , so that  $\mathcal{A}_{\mathcal{I}} = \{H_\alpha, H_\gamma\}$  is of rank 2. We consider two different maximal parabolic subsystems in turn.

(a). First let  $\Phi_0$  be the subsystem of  $\Phi$  of type  $A_2$  generated by  $\alpha$  and  $\beta$ . Then  $\mathcal{I}_0 = \{\beta, \alpha + \beta\}$  and  $\Phi_0^c \cap \mathcal{I}^c = \{\gamma\}$ . So,  $\mathcal{I}_0^c = \{\alpha\}$  and  $\mathcal{A}_{\mathcal{I}_0} = \{H_\alpha\}$ . Clearly,  $\mathcal{I}_0$  is not an ideal in  $\Phi^+$ . Note that  $X_0 = H_\alpha \cap H_\beta \notin L(\mathcal{A}_{\mathcal{I}})$ . But  $(\mathcal{A}_{\mathcal{I}})_{X_0} = \{H_\alpha\}$  so that  $r((\mathcal{A}_{\mathcal{I}})_{X_0}) = r(\mathcal{A}_{\mathcal{I}}) - 1$ . In particular, Condition 1.10 holds in this instance.

(b). Second let  $\Phi_0$  be the subsystem of  $\Phi$  of type  $A_1^2$  generated by  $\alpha$  and  $\gamma$ . This time  $\mathcal{I}_0 = \emptyset$  which is of course an ideal in  $\Phi^+$ . Moreover, we have  $\mathcal{I}_0^c = \Phi_0^+ = \{\alpha, \gamma\} = \mathcal{I}^c$ . Now  $X_0 = H_\alpha \cap H_\gamma \in L(\mathcal{A}_{\mathcal{I}})$ . So that  $\mathcal{A}_{\mathcal{I}_0} = (\mathcal{A}_{\mathcal{I}})_{X_0} = \mathcal{A}_{\mathcal{I}}$ , and in particular,  $r(\mathcal{A}_{\mathcal{I}_0}) = r(\mathcal{A}_{\mathcal{I}})$ . Here Condition 1.10 is not satisfied, as  $\Phi_0^c \cap \mathcal{I}^c = \emptyset$ .

Our next observation, showing that Condition 1.10 entails the presence of a modular element in  $L(\mathcal{A}_{\mathcal{I}})$  of rank  $r(\mathcal{A}_{\mathcal{I}}) - 1$ , is pivotal for our entire analysis.

**Lemma 3.4.** *If  $\mathcal{I} \subseteq \Phi^+$  and  $\Phi_0$  satisfy Condition 1.10, then for  $X_0 := \cap_{\gamma \in \Phi_0^+} H_\gamma$ , the center*

$$Z := T((\mathcal{A}_{\mathcal{I}})_{X_0})$$

*of  $(\mathcal{A}_{\mathcal{I}})_{X_0}$  is modular of rank  $r(\mathcal{A}_{\mathcal{I}}) - 1$  in  $L(\mathcal{A}_{\mathcal{I}})$ .*

*Proof.* By construction and Lemma 3.1, we have

$$(3.5) \quad \mathcal{A}_{\mathcal{I}_0} = (\mathcal{A}_{\mathcal{I}})_{X_0} = (\mathcal{A}_{\mathcal{I}})_Z.$$

Thanks to Condition 1.10,  $\Phi_0^c \cap \mathcal{I}^c \neq \emptyset$  and  $\Phi_0^c \cap \mathcal{I}^c$  is linearly ordered by height. Consequently, there is a unique simple root  $\alpha$  in  $\Phi_0^c \cap \mathcal{I}^c$ . Since  $\alpha$  is linearly independent from  $\mathcal{I}_0^c$  and every root in  $\mathcal{I}^c$  is a  $\mathbb{Z}_{\geq 0}$ -linear combination of  $\alpha$  and roots from  $\mathcal{I}_0^c$ , we have  $r(\mathcal{A}_{\mathcal{I}_0}) = r(\mathcal{A}_{\mathcal{I}}) - 1$ . We conclude that  $r(Z) = r((\mathcal{A}_{\mathcal{I}})_Z) = r(\mathcal{A}_{\mathcal{I}}) - 1$ , by (3.5).

It remains to show that  $Z$  is modular. If  $\alpha \neq \beta \in \Phi_0^c \cap \mathcal{I}^c$ , then by Condition 1.10, there is a  $\gamma \in \Phi_0^+$  so that  $\alpha, \beta$  and  $\gamma$  are linearly dependent. Since  $\alpha$  and  $\beta$  both belong to  $\mathcal{I}^c$ , so does  $\gamma$ . It follows that  $\gamma \in \Phi_0^+ \cap \mathcal{I}^c = \mathcal{I}_0^c$ . Thus  $H_\gamma \in \mathcal{A}_{\mathcal{I}_0} = (\mathcal{A}_{\mathcal{I}})_Z$ , by (3.5). On the other hand,  $H_\alpha, H_\beta \in \mathcal{A}_{\mathcal{I}} \setminus (\mathcal{A}_{\mathcal{I}})_Z$ . Since  $\alpha, \beta$  and  $\gamma$  are linearly dependent,  $Z$  is modular, by Lemma 2.1.  $\square$

**Lemma 3.6.** *Suppose  $\Phi, \mathcal{I}$  and  $\Phi_0$  satisfy Condition 1.10. Let  $\delta$  be in  $\Phi_0^c \cap \mathcal{I}^c$ . Then the restriction of  $\mathcal{A}_{\mathcal{I}}$  to  $H_\delta$  is isomorphic to the arrangement of ideal type  $\mathcal{A}_{\mathcal{I}_0}$  in  $\mathcal{A}(\Phi_0)$ .*

*Proof.* By Lemma 3.4, for  $X_0 := \cap_{\gamma \in \Phi_0^+} H_\gamma$ , the center  $Z := T((\mathcal{A}_{\mathcal{I}})_{X_0})$  of  $(\mathcal{A}_{\mathcal{I}})_{X_0}$  is modular of rank  $r(\mathcal{A}_{\mathcal{I}}) - 1$  in  $L(\mathcal{A}_{\mathcal{I}})$ . By (3.5),  $\mathcal{A}_{\mathcal{I}_0} = (\mathcal{A}_{\mathcal{I}})_{X_0} = (\mathcal{A}_{\mathcal{I}})_Z$ . Therefore, it follows from Lemma 2.2 that the restriction map  $\varrho_\delta : \mathcal{A}_{\mathcal{I}_0} \rightarrow \mathcal{A}_{\mathcal{I}}^{H_\delta}$  given by  $H_\gamma \mapsto H_\delta \cap H_\gamma$  defines an isomorphism between  $\mathcal{A}_{\mathcal{I}_0}$  and  $\mathcal{A}_{\mathcal{I}}^{H_\delta}$ .  $\square$

We are now in a position to prove Theorem 1.12.

*Proof of Theorem 1.12.* The reverse implications follow in each instance from Corollary 3.2.

Now consider the forward implications. If  $\mathcal{A}_{\mathcal{I}}$  has rank at most 2, then all statements clearly hold, as then  $\mathcal{A}_{\mathcal{I}}$  is supersolvable, cf. Theorems 2.12 and 2.15 and Proposition 2.20. So suppose that  $r(\mathcal{A}_{\mathcal{I}}) \geq 3$ .

If  $\Phi_0^c \cap \mathcal{I}^c = \emptyset$ , then  $\mathcal{A}_{\mathcal{I}}$  is the product of the 1-dimensional empty arrangement  $\emptyset_1$  and  $\mathcal{A}_{\mathcal{I}_0}$ . So each statement follows from Propositions 2.14, 2.9, and 2.26, respectively. Therefore, we may assume that  $\Phi_0^c \cap \mathcal{I}^c \neq \emptyset$ . Then define the non-empty subarrangement

$$\mathcal{A}_{\mathcal{I}_0}^c := \{H_\beta \mid \beta \in \Phi_0^c \cap \mathcal{I}^c\} = \mathcal{A}_{\mathcal{I}} \setminus \mathcal{A}_{\mathcal{I}_0}$$

of  $\mathcal{A}_{\mathcal{I}}$ . We may thus decompose  $\mathcal{A}_{\mathcal{I}}$  as the proper disjoint union

$$\mathcal{A}_{\mathcal{I}} = \mathcal{A}_{\mathcal{I}_0} \coprod \mathcal{A}_{\mathcal{I}_0}^c.$$

Now Condition 1.10 implies that the center  $Z := T((\mathcal{A}_{\mathcal{I}})_{X_0})$  of  $(\mathcal{A}_{\mathcal{I}})_{X_0}$  is modular of rank  $r(\mathcal{A}_{\mathcal{I}}) - 1$  in  $L(\mathcal{A}_{\mathcal{I}})$ , by Lemma 3.4. Thus the forward implications follow thanks to (3.5) from Corollary 2.13 and Lemmas 2.7 and 2.24, respectively.  $\square$

Armed with Theorem 1.12 we can now derive Theorems 1.13 and 1.5.

*Proof of Theorem 1.13.* If  $\mathcal{A}_{\mathcal{I}}$  is reducible, then  $\mathcal{A}_{\mathcal{I}}$  is the product of smaller rank arrangements of ideal type. The result then follows from the inductive hypothesis along with Proposition 2.9.

If  $\mathcal{A}_{\mathcal{I}}$  is irreducible and there is a maximal parabolic subsystem of  $\Phi$  such that Condition 1.10 is satisfied, it follows from the inductive hypothesis and Theorem 1.12(ii) that  $\mathcal{A}_{\mathcal{I}}$  is inductively free.  $\square$

We now apply Theorem 1.12 to various types.

In [ST06, Lem. 7.1], Sommers and Tymoczko showed that for  $\Phi$  of type  $A_n$ ,  $B_n$  or  $C_n$  and the canonical choice of maximal subsystem  $\Phi_0$  of type  $A_{n-1}$ ,  $B_{n-1}$  or  $C_{n-1}$ , respectively, Condition 1.10 is satisfied for any ideal  $\mathcal{I}$  with  $\Phi_0^c \cap \mathcal{I}^c \neq \emptyset$ .

*Proof of Theorem 1.5.* For  $\Phi$  of rank 2, the result follows by the first part of Theorem 2.12.

Let  $\Phi$  be of type  $A_n$ ,  $B_n$ , or  $C_n$  for  $n \geq 3$  and let  $\mathcal{A} = \mathcal{A}(\Phi)$  be the Weyl arrangement of  $\Phi$ . We argue by induction on  $n$  and suppose that the result holds for smaller rank root systems of the same type as  $\Phi$ . Let  $\Phi_0$  be the standard maximal parabolic subsystem of  $\Phi$  of type  $A_{n-1}$ ,  $B_{n-1}$  or  $C_{n-1}$ , respectively. If  $\Phi_0^c \cap \mathcal{I}^c = \emptyset$ , then  $\mathcal{A}_{\mathcal{I}}$  is the product of the 1-dimensional empty arrangement  $\emptyset_1$  and an arrangement of ideal type of  $\Phi_0$ , and the latter is supersolvable by induction on  $n$ , and so the result follows from Proposition 2.14.

Therefore, we may assume that  $\Phi_0^c \cap \mathcal{I}^c \neq \emptyset$ . It follows from [ST06, §7] that  $\Phi_0^c = \Phi^+ \setminus \Phi_0^+$  is linearly ordered by height and that for each  $\mathcal{I}$  with  $\Phi_0^c \cap \mathcal{I}^c \neq \emptyset$ , Condition 1.10 is satisfied. It thus follows by induction on  $n$  and Theorem 1.12(i) that each  $\mathcal{A}_{\mathcal{I}}$  is supersolvable.  $\square$

With Theorem 1.5 we readily get further instances of supersolvable arrangements of ideal type in other types as well.

**Example 3.7.** Let  $\Phi$  be a reduced root system and let  $\mathcal{I}$  be an ideal in  $\Phi^+$  with  $\mathcal{I} \cap \Pi \neq \emptyset$ . Suppose that the simple factors in the complement  $\Pi \setminus \mathcal{I}$  are all of type  $A$ ,  $B$  or  $C$ . Then  $\mathcal{A}_{\mathcal{I}}$  is a product of arrangements of ideal type of types  $A$ ,  $B$  or  $C$ . So  $\mathcal{A}_{\mathcal{I}}$  is supersolvable, thanks to Theorem 1.5 and Proposition 2.14. For instance this is the case for any such  $\mathcal{I}$  in case  $\Phi$  is of type  $F_4$ .

Moreover, if it is the case that  $\mathcal{A}_{\mathcal{I}_0}$  is supersolvable in all instances when Condition 1.10 holds, then also  $\mathcal{A}_{\mathcal{I}}$  is supersolvable, by Theorem 1.12(i). This is also the case for  $F_4$ , as then  $\Phi_0$  is of type  $B_3$  or  $C_3$ . Consequently, out of the total of 105 ideals in type  $F_4$  the 83 instances covered in Theorem 1.14 (i) and (ii) are supersolvable.

Here is a uniform example of a supersolvable arrangement of ideal type in every type.

**Example 3.8.** Observe that  $\mathcal{I}_2^c = \Pi$  and so  $\mathcal{A}_{\mathcal{I}_2}$  is the Boolean arrangement of rank  $n$  which is known to be supersolvable, cf. [OT92, Ex. 2.33].

Next we consider the case when  $\Phi$  is of type  $D_n$ . Here and later on we use the notation for the positive roots from [Bou68, Planche IV].



*Proof of Theorem 1.7.* Let  $\Phi$  be of type  $D_n$ , for  $n \geq 4$  and let  $\Phi_0$  be the standard subsystem of  $\Phi$  of type  $D_{n-1}$ . Then  $\Phi_0^c = \{e_1 \pm e_j \mid 2 \leq j \leq n\}$ . Note that  $\Phi_0^c$  is not linearly ordered by  $\preceq$ , for  $\beta^\pm := \beta_n^\pm := e_1 \pm e_n$  both have height  $n - 1$ .

We argue by induction on  $n$ . For  $n = 3$ ,  $\Phi$  is of type  $D_3 = A_3$ , and so the result follows from Theorems 1.5 and 2.15.

Now suppose that  $n \geq 4$  and that the result holds for smaller rank root systems of type  $D$ . If  $\Phi_0^c \cap \mathcal{I}^c = \emptyset$ , then  $\mathcal{A}_{\mathcal{I}}$  is the product of the 1-dimensional empty arrangement  $\emptyset_1$  and an arrangement of ideal type of the subsystem of type  $D_{n-1}$ . Since the latter is inductively free by induction on the rank, the result follows from Proposition 2.9. Therefore, we may assume that  $\Phi_0^c \cap \mathcal{I}^c \neq \emptyset$ .

(a). Suppose first that  $\mathcal{I}$  is an ideal in  $\Phi^+$  so that Condition 1.10 is satisfied. This is precisely the case as long as not both of  $\beta^\pm = e_1 \pm e_n$  belong to  $\mathcal{I}^c$ . Let  $\delta$  be the unique root of maximal height in  $\Phi_0^c \cap \mathcal{I}^c$  and set  $\mathcal{I}_0 = \mathcal{I} \cap \Phi_0$ . Then  $\mathcal{I}_0$  is an ideal in  $\Phi_0^+$ . Set  $m_s^{\mathcal{I}} := \text{ht}(\delta)$ , so that thanks to Condition 1.10, we have

$$m_s^{\mathcal{I}} = |\Phi_0^c \cap \mathcal{I}^c| = |\mathcal{A}_{\mathcal{I}} \setminus \mathcal{A}_{\mathcal{I}_0}|.$$

By induction on the rank,  $\mathcal{A}_{\mathcal{I}_0}$  is inductively free and by construction  $m_1^{\mathcal{I}}, \dots, m_{k-1}^{\mathcal{I}}$  are the non-zero exponents of  $\mathcal{A}_{\mathcal{I}_0}$ , where  $m_1^{\mathcal{I}}, \dots, m_s^{\mathcal{I}}$  are the ideal exponents of  $\mathcal{I}$ . Note that here we do not partially order the ideal exponents as in (1.2).

It follows from Theorem 1.12(ii) that  $\mathcal{A}_{\mathcal{I}}$  is inductively free with exponents  $\exp \mathcal{A}_{\mathcal{I}} = \{0^{n-k}, m_1^{\mathcal{I}}, \dots, m_s^{\mathcal{I}}\}$ , as desired.

(b). Now we consider the cases when both  $\beta^\pm = e_1 \pm e_n$  do belong to  $\mathcal{I}^c$ . Here we follow closely the proof of [ST06, Thm. 11.1].

Suppose first that  $\mathcal{I}$  is such that both  $\beta^\pm$  are maximal in  $\Phi_0^c \cap \mathcal{I}^c$  with respect to  $\preceq$ . Set  $\mathcal{I}^+ := \mathcal{I} \cup \{\beta^+\}$ . Then by case (a) proved above,  $\mathcal{A}_{\mathcal{I}^+}$  is inductively free and

$$\exp \mathcal{A}_{\mathcal{I}^+} = \{m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1, n-2\}.$$

Let  $(\mathcal{A}_{\mathcal{I}}, \mathcal{A}_{\mathcal{I}}' = \mathcal{A}_{\mathcal{I}^+}, \mathcal{A}_{\mathcal{I}}'')$  be the triple of  $\mathcal{A}_{\mathcal{I}}$  with respect to  $H_{\beta^+}$ . It remains to show that  $\mathcal{A}_{\mathcal{I}}^{H_{\beta^+}}$  is inductively free with

$$\exp \mathcal{A}_{\mathcal{I}}^{H_{\beta^+}} = \{m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1\}.$$

It then follows from Theorem 2.3 that  $\mathcal{A}_{\mathcal{I}}$  is inductively free with

$$\exp \mathcal{A}_{\mathcal{I}} = \{m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1, n-1\}.$$

In order to show that  $\mathcal{A}_{\mathcal{I}}^{H_{\beta^+}}$  is inductively free with the desired exponents, we argue as follows. Consider the triple of  $\mathcal{A}_{\mathcal{I}}^{H_{\beta^+}}$  with respect to  $H_{\beta^+} \cap H_{\beta^-}$ . Then the deleted arrangement  $\mathcal{A}_{\mathcal{I}}^{H_{\beta^+}} \setminus \{H_{\beta^+} \cap H_{\beta^-}\}$  coincides with  $\mathcal{A}_{\mathcal{I}^-}^{H_{\beta^+}}$ , where  $\mathcal{I}^- := \mathcal{I} \cup \{\beta^-\}$  and so the latter is inductively free with exponents  $\{m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-2\}$ , by case (a) above. For  $\mathcal{I}^-$  satisfies Condition 1.10 above and  $\beta^+$  is maximal in  $\Phi_0^c \cap (\mathcal{I}^-)^c$ . Finally, we need to show that the restricted arrangement  $\left(\mathcal{A}_{\mathcal{I}}^{H_{\beta^+}}\right)^{H_{\beta^+} \cap H_{\beta^-}}$  is inductively free. Since  $H_{\beta^+} \cap H_{\beta^-}$  coincides with the intersection of the null spaces of  $e_1$  and  $e_n$ , arguing as in the proof of [ST06, Thm. 11.1], this restricted arrangement coincides with an arrangement of ideal type in a root system of type  $B_{n-2}$  with exponents given by  $\{m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}\}$ . Thanks to Theorems 1.5 and 2.15, the latter is inductively free. So,  $\mathcal{A}_{\mathcal{I}}^{H_{\beta^+}}$  is inductively free, by Theorem 2.3.

Finally, we consider the case when  $\delta = e_1 + e_{2n-1-l}$  is the maximal element of  $\Phi_0^c \cap \mathcal{I}^c$  for some  $l > n - 1$ . We argue as in the proof of [ST06, Thm. 11.1] and deduce again that  $\mathcal{A}_{\mathcal{I}}$  is inductively free with

$$\exp \mathcal{A}_{\mathcal{I}} = \{m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1, l\},$$

as follows. By induction on  $|\mathcal{I}^c|$ , we know that  $\mathcal{A}_{\mathcal{I}'}$  is inductively free, for  $\mathcal{I}' = \mathcal{I} \cup \{\delta\}$ , with

$$\exp \mathcal{A}_{\mathcal{I}'} = \{m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1, l-1\}.$$

By the argument from the proof of [ST06, Thm. 11.1], the restricted arrangement  $\mathcal{A}_{\mathcal{I}}^{H_{\delta}}$  is isomorphic to  $\mathcal{A}_{\mathcal{I}}^{H_{\beta^+}}$  in the case above, where both  $\beta^{\pm}$  were maximal in  $\Phi_0^c \cap \mathcal{I}^c$ . From that case we infer that  $\mathcal{A}_{\mathcal{I}}^{H_{\beta^+}}$  is inductively free, thus so is  $\mathcal{A}_{\mathcal{I}}^{H_{\delta}}$  with

$$\exp \mathcal{A}_{\mathcal{I}}^{H_{\delta}} = \{m_1^{\mathcal{I}}, \dots, m_{n-2}^{\mathcal{I}}, n-1\}.$$

It follows from Theorem 2.3 that also in this case  $\mathcal{A}_{\mathcal{I}}$  is inductively free.  $\square$

**Example 3.9.** Theorem 1.12(i) can also be used to show that there are arrangements of ideal type which are supersolvable in type  $D_n$  beyond the cases treated in Theorem 1.31(i). For instance let  $\mathcal{I}$  be the ideal generated by  $e_{n-3} + e_n$ ,  $e_{n-3} - e_n$ , or  $e_{n-2} + e_{n-1}$ . Let  $\Phi_0$  be the standard subsystem of type  $A_{n-1}$  so that the generator of  $\mathcal{I}$  does not belong to  $\Phi_0$  (there are two choices in the last instance). Then one easily checks that Condition 1.10 is satisfied. The fact that  $\mathcal{A}_{\mathcal{I}}$  is supersolvable then follows from Theorems 1.5 and 1.12(i).

Next we give a uniform argument for the penultimate ideal  $\mathcal{I}_{h-1} = \{\theta\}$  in all cases.

*Proof of Theorem 1.9.* Let  $\mathcal{A} = \mathcal{A}(W)$  be the Weyl arrangement of  $W$ . Consider the triple  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  with respect to  $H_{\theta}$ . In case there is only one root length  $W$  is transitive on  $\Phi$ . In the other instances (i.e. for  $B_n$  and  $F_4$ ) the restrictions of  $\mathcal{A}$  to hyperplanes with respect to a short and long root are isomorphic, cf. [OT92, Prop. 6.82, Table C.9]. Therefore, since  $\mathcal{A}$  is inductively free,  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  is a triple of inductively free arrangements. In particular,  $\mathcal{A}' = \mathcal{A}_{\mathcal{I}_{h-1}}$  is inductively free.  $\square$

**Remark 3.10.** Lemma 3.6 and Theorem 1.12(ii) can also be employed in the exceptional instances as well. Suppose  $\mathcal{I}$  is an ideal in  $\Phi^+$  such that Condition 1.10 is fulfilled for some fixed choice of a maximal parabolic subsystem  $\Phi_0$ . Let  $\delta$  be the unique root of maximal height in  $\Phi_0^c \cap \mathcal{I}^c$  and  $\mathcal{I}_0 = \mathcal{I} \cap \Phi_0^+$ . Consider the restriction of  $\mathcal{A}_{\mathcal{I}}$  with respect to  $H_{\delta}$ . In our applications in Section 4 we argue by induction on the rank of the underlying root system, so that the arrangement of ideal type  $\mathcal{A}_{\mathcal{I}_0}$  of smaller rank is inductively free. Then using Lemma 3.6, Theorems 1.5 and 1.7, it follows that  $\mathcal{A}_{\mathcal{I}}^{H_{\delta}} \cong \mathcal{A}_{\mathcal{I}_0}$  is inductively free. It thus follows from Theorem 1.12(ii) that also  $\mathcal{A}_{\mathcal{I}}$  is inductively free with the desired exponents. We illustrate this in the following example.

**Example 3.11.** Let  $\Phi$  be of type  $F_4$ . We use the notation for the roots in  $\Phi$  as in [Bou68, Planche VIII]. We consider two examples of ideals. In our first case let  $\mathcal{I}$  be the ideal generated by 1120 and 0122. Let  $\Phi_0$  be the standard subsystem of type  $C_3$ . Then  $\mathcal{I}^c = \{\underline{1000}, 0100, 0010, 0001, \underline{1100}, 0110, 0011, \underline{1110}, 0120, 0111, \underline{1111}, 0121\}$ , where the roots in  $\Phi_0^c \cap \mathcal{I}^c$  are underlined. It is easy to check that  $\mathcal{I}$  and  $\Phi_0$  satisfy Condition 1.10.

In our second example let  $\mathcal{I}$  be the ideal generated by 0121. This time let  $\Phi_0$  be the standard subsystem of type  $B_3$ . Then  $\mathcal{I}^c = \{1000, 0100, 0010, \underline{0001}, 1100, 0110, \underline{0011}, 1110,$

$0120, \underline{0111}, 1120, \underline{1111}, 1220\}$ , where the roots in  $\Phi_0^c \cap \mathcal{I}^c$  are underlined. It is again easy to check that  $\mathcal{I}$  and  $\Phi_0$  satisfy Condition 1.10.

Here the subsystems used must not be interchanged; for  $\Phi_0$  of type  $B_3$ , the first ideal does not satisfy Condition 1.10 and likewise neither does the second for  $\Phi_0$  of type  $C_3$ .

It follows from Theorems 1.5 and 1.12(i) that both  $\mathcal{A}_{\mathcal{I}}$  are supersolvable and that the exponents are  $\{1, 3, 4, 4\}$  and  $\{1, 3, 4, 5\}$ , respectively; cf. Example 3.7.

We investigate the inductively free  $\mathcal{A}_{\mathcal{I}}$  in the exceptional instances in more detail in §4.

#### 4. INDUCTIVELY FREE $\mathcal{A}_{\mathcal{I}}$ FOR $\Phi$ OF EXCEPTIONAL TYPE

4.1. Thanks to Proposition 2.9, Theorem 1.14 readily reduces to the case when  $\Phi$  is irreducible which we assume from now on. Let  $\mathcal{A} = \mathcal{A}(\Phi)$  be the Weyl arrangement for  $\Phi$  irreducible of exceptional type. Since any arrangement of rank at most two is inductively free, we may suppose that  $W$  has rank at least 4.

We use the labeling of the Dynkin diagram of  $\Phi$  and the notation for roots in  $\Phi$  from [Bou68, Planche V - VIII]. We argue by induction on the rank of the underlying root system and therefore assume that each arrangement of ideal type is inductively free for root systems of smaller rank. So fix  $\Phi$  and let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . Arguing further by induction on  $|\mathcal{I}^c|$ , we may assume that  $\mathcal{A}_{\mathcal{J}}$  is inductively free for every ideal  $\mathcal{J}$  properly containing  $\mathcal{I}$ .

4.2. Our strategy is to consider all ideals  $\mathcal{I}$  such that  $\mathcal{I} \subseteq \mathcal{I}_t$  but  $\mathcal{I} \not\subseteq \mathcal{I}_{t+1}$  for successive values of  $t \geq 1$ . This means each such ideal contains a root of height  $t$  but no roots of smaller height. Then we determine all instances when Condition 1.10 is satisfied for each such  $\mathcal{I}$  for a suitable choice of subsystem  $\Phi_0$  of  $\Phi$  in Tables 6 – 9 below. In each of these tables we list for a given root  $\beta$  in  $\mathcal{I}$  of height  $t$  a parabolic subsystem  $\Phi_0$  of  $\Phi$  and the resulting set of roots in  $\Phi_0^c \cap \mathcal{I}^c$  relevant for Condition 1.10. Here we determine  $\Phi_0^c \cap \mathcal{I}^c$  under the assumption that  $\beta$  is the only root of height  $t$  in  $\mathcal{I}$ . In case there are additional roots of height  $t$  in  $\mathcal{I}$ , the set  $\Phi_0^c \cap \mathcal{I}^c$  may be smaller but still satisfies Condition 1.10. In each case we know by induction that  $\mathcal{A}_{\mathcal{I}_0}$  is inductively free, so that we can conclude from Theorem 1.12(ii) that  $\mathcal{A}_{\mathcal{I}}$  is also inductively free.

For instance, consider the next to last entry for  $E_7$  in Table 6. Here  $\mathcal{I} = \left\langle \begin{smallmatrix} 001110 \\ 0 \end{smallmatrix} \right\rangle$ , and for  $\Phi_0$  of type  $E_6$  it follows that  $\mathcal{I}_0 = \left\langle \begin{smallmatrix} 00111 \\ 0 \end{smallmatrix} \right\rangle$ . The fact that  $\mathcal{A}_{\mathcal{I}_0}$  is inductively free follows from the last row for  $E_6$  of the same table. So  $\mathcal{A}_{\mathcal{I}}$  is inductively free by induction and Theorem 1.12(ii). For another example, consider the last case for  $E_7$  in Table 6. Here  $\mathcal{I} = \left\langle \begin{smallmatrix} 000111 \\ 0 \end{smallmatrix} \right\rangle$ , and for  $\Phi_0$  of type  $E_6$  we have  $\mathcal{I}_0^c = \Phi_0^+$ , so that  $\mathcal{A}_{\mathcal{I}_0} = \mathcal{A}(E_6)$  which is inductively free. So then again  $\mathcal{A}_{\mathcal{I}}$  is inductively free thanks to Theorem 1.12(ii).

4.3. Returning to Table 5, the last entry in each column above the last row labeled by  $*$  indicates that for every ideal  $\mathcal{I}$  with  $\mathcal{I} \subseteq \mathcal{I}_t$ , there is no maximal parabolic subsystem  $\Phi_0$  so that Condition 1.10 is satisfied. Naturally, if  $\mathcal{I}$  is small, then  $\mathcal{I}^c$  is large. So that for sufficiently small  $\mathcal{I}$  Condition 1.10 fails simply because  $\Phi_0^c \cap \mathcal{I}^c$  is no longer linear for any choice of maximal subsystem  $\Phi_0$ . So for instance, if  $\Phi$  is of type  $F_4$ , then this is the case for all  $\mathcal{I}$  whose roots have height at least 5, and according to the entry in Table 5, there are exactly 22 such instances. The final row labeled by  $*$  gives the total number of all ideals  $\mathcal{I}$  where Condition 1.10 fails for every choice of a maximal subsystem  $\Phi_0$  of  $\Phi$ . From the list

of these cases we have removed the 2 instances corresponding to the full Weyl arrangement  $\mathcal{A}_{\mathcal{I}_h} = \mathcal{A}(\Phi)$  and the case of the penultimate ideal  $\mathcal{A}_{\mathcal{I}_{h-1}}$  which are both inductively free.

4.4. Now if  $\mathcal{I} \cap \Pi \neq \emptyset$ , i.e. if  $\mathcal{I}$  is not strictly positive, then  $\mathcal{I}^c$  is either the complement of an ideal in a smaller rank root system or is the product of complements of ideals in direct products of smaller rank root systems. It therefore follows from our induction hypothesis and Proposition 2.9 that also  $\mathcal{A}_{\mathcal{I}}$  is inductively free in this instance. We may therefore assume that  $\mathcal{I}$  is strictly positive, i.e.  $\mathcal{I} \subseteq \mathcal{I}_2$ . Observe that  $\mathcal{I}_2^c = \Pi$  and so  $\mathcal{A}_{\mathcal{I}_2}$  is inductively free, see Example 3.8.

4.5. Next suppose that  $\mathcal{I} \subseteq \mathcal{I}_2$  and  $\mathcal{I} \not\subseteq \mathcal{I}_3$ , i.e.  $\mathcal{I}$  contains a root of height 2. Then  $\mathcal{A}_{\mathcal{I}}$  is again the product of two arrangements of ideal type of smaller rank root systems and so the result follows from our induction hypothesis and Proposition 2.9. We therefore may assume that  $\mathcal{I} \subseteq \mathcal{I}_3$ .

4.6. Next we consider the case when  $\mathcal{I} \subseteq \mathcal{I}_3$  and  $\mathcal{I} \not\subseteq \mathcal{I}_4$ . Then there is a root of height 3 in  $\mathcal{I}$  but no root of smaller height. One readily checks that there is always a suitable maximal subsystem  $\Phi_0$  in each case so that Condition 1.10 is satisfied. We list the various cases in Table 6 below. For each fixed root  $\beta$  of height 3 we consider the case  $\mathcal{I} = \langle \beta \rangle$  and list a suitable subsystem  $\Phi_0$  such that Condition 1.10 is fulfilled. This is easily checked in each instance. In type  $E_6$  there are two standard subsystems of type  $D_5$  which are conjugate by the graph automorphism of  $E_6$ . In order to distinguish between them, here and later on,  $D_5$ , respectively  $D'_5$ , denotes the one complementary to  $\alpha_1$ , respectively  $\alpha_6$ .

We should point out that in case  $\mathcal{I}$  contains additional roots of height 3, then Condition 1.10 is still fulfilled albeit  $\Phi_0^c \cap \mathcal{I}^c$  may be a proper subset of the one shown in Table 6. For instance, if  $\Phi$  is of type  $E_6$  and  $\mathcal{I} = \left\langle \begin{smallmatrix} 01100 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 00110 \\ 1 \end{smallmatrix} \right\rangle$ , then for  $\Phi_0$  of type  $A_5$  as in the second row for  $E_6$  in Table 6,  $\Phi_0^c \cap \mathcal{I}^c$  only consists of  $\alpha_2$  and  $\alpha_2 + \alpha_4$ .

It follows by induction on the rank and the results from §§4.4 - 4.6 that  $\mathcal{A}_{\mathcal{I}}$  is inductively free for any  $\mathcal{I}$  with  $\mathcal{I} \not\subseteq \mathcal{I}_4$ .

4.7. Next, we consider the case when  $\mathcal{I} \subseteq \mathcal{I}_4$  and  $\mathcal{I} \not\subseteq \mathcal{I}_5$ . Then there is a root of height 4 in  $\mathcal{I}$  but no root of smaller height. In Table 7, for each fixed root  $\beta$  of height 4 we first consider the case  $\mathcal{I} = \langle \beta \rangle$  and – provided there exists one – list a suitable maximal parabolic subsystem  $\Phi_0$  such that Condition 1.10 holds. This is then easy to verify in each instance.

There are some ideals  $\mathcal{I}$  of this kind when there is no maximal standard parabolic subsystem  $\Phi_0$  such that Condition 1.10 is fulfilled. This is indicated with the label “ $\times$ ” in the corresponding row. For instance, this occurs when  $\Phi$  is of type  $E_6$  and  $\mathcal{I} = \left\langle \begin{smallmatrix} 01110 \\ 1 \end{smallmatrix} \right\rangle$ . If there are additional roots of height 4 in  $\mathcal{I}$ , then we can ensure again that Condition 1.10 holds. E.g., let  $\mathcal{I} = \left\langle \begin{smallmatrix} 11110 \\ 0 \end{smallmatrix}, \begin{smallmatrix} 01110 \\ 1 \end{smallmatrix} \right\rangle$ . Then for  $\Phi_0$  of type  $D_5$ ,  $\Phi_0^c \cap \mathcal{I}^c$  is the same set as in the first row for  $E_6$  in Table 7. One checks that  $\mathcal{I} = \left\langle \begin{smallmatrix} 01110 \\ 1 \end{smallmatrix} \right\rangle$  and  $\mathcal{I} = \left\langle \begin{smallmatrix} 01110 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 11111 \\ 0 \end{smallmatrix} \right\rangle$  are the only ideals of this nature for which Condition 1.10 fails for any choice of a maximal standard subsystem  $\Phi_0$  of  $\Phi$ . Likewise, for  $\Phi$  of type  $E_7$ , respectively of type  $E_8$ , there are 3, respectively 8, ideals of this kind for which Condition 1.10 does not hold for any choice of a maximal

$\Phi$	$\beta \in \mathcal{I}$	$\Phi_0$	$\Phi_0^c \cap \mathcal{I}^c$
$F_4$	1110	$C_3$	1000, 1100
	0120	$C_3$	1000, 1100, 1110, 1111
	0111	$B_3$	0001, 0011
$E_6$	$\begin{smallmatrix} 11100 \\ 0 \end{smallmatrix}$	$D_5$	$\begin{smallmatrix} 10000 & 11000 \\ 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 01100 \\ 1 \end{smallmatrix}$	$A_5$	$\begin{smallmatrix} 00000 & 00100 & 00110 & 00111 \\ 1 & , & 1 & , & 1 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 01110 \\ 0 \end{smallmatrix}$	$D_5$	$\begin{smallmatrix} 10000 & 11000 & 11100 & 11100 \\ 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 00110 \\ 1 \end{smallmatrix}$	$A_5$	$\begin{smallmatrix} 00000 & 00100 & 01100 & 11100 \\ 1 & , & 1 & , & 1 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 00111 \\ 0 \end{smallmatrix}$	$D'_5$	$\begin{smallmatrix} 00001 & 00011 \\ 0 & , & 0 \end{smallmatrix}$
$E_7$	$\begin{smallmatrix} 111000 \\ 0 \end{smallmatrix}$	$D_6$	$\begin{smallmatrix} 100000 & 110000 \\ 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 011000 \\ 1 \end{smallmatrix}$	$A_6$	$\begin{smallmatrix} 000000 & 001000 & 001100 & 001110 & 001111 \\ 1 & , & 1 & , & 1 & , & 1 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 011100 \\ 0 \end{smallmatrix}$	$D_6$	$\begin{smallmatrix} 100000 & 110000 & 111000 & 111000 \\ 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 001100 \\ 1 \end{smallmatrix}$	$A_6$	$\begin{smallmatrix} 000000 & 001000 & 011000 & 111000 \\ 1 & , & 1 & , & 1 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 001110 \\ 0 \end{smallmatrix}$	$E_6$	$\begin{smallmatrix} 000001 & 000011 & 000111 \\ 0 & , & 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 000111 \\ 0 \end{smallmatrix}$	$E_6$	$\begin{smallmatrix} 000001 & 000011 \\ 0 & , & 0 \end{smallmatrix}$
$E_8$	$\begin{smallmatrix} 1110000 \\ 0 \end{smallmatrix}$	$D_7$	$\begin{smallmatrix} 1000000 & 1100000 \\ 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 0110000 \\ 1 \end{smallmatrix}$	$A_7$	$\begin{smallmatrix} 0000000 & 0010000 & 0011000 & 0011100 & 0011110 & 0011111 \\ 1 & , & 1 & , & 1 & , & 1 & , & 1 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 0111000 \\ 0 \end{smallmatrix}$	$D_7$	$\begin{smallmatrix} 1000000 & 1100000 & 1110000 & 1110000 \\ 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 0011000 \\ 1 \end{smallmatrix}$	$A_7$	$\begin{smallmatrix} 0000000 & 0010000 & 0110000 & 1110000 \\ 1 & , & 1 & , & 1 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 0011100 \\ 0 \end{smallmatrix}$	$E_7$	$\begin{smallmatrix} 0000001 & 0000011 & 0000111 & 0001111 \\ 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 0001110 \\ 0 \end{smallmatrix}$	$E_7$	$\begin{smallmatrix} 0000001 & 0000011 & 0000111 \\ 0 & , & 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 0000111 \\ 0 \end{smallmatrix}$	$E_7$	$\begin{smallmatrix} 0000001 & 0000011 \\ 0 & , & 0 \end{smallmatrix}$

TABLE 6. Condition 1.10 for  $\mathcal{I} \subseteq \mathcal{I}_3$  and  $\mathcal{I} \not\subseteq \mathcal{I}_4$

standard parabolic subsystem  $\Phi_0$  of  $\Phi$ . For instance, the ones in type  $E_7$  are  $\mathcal{I} = \left\langle \begin{smallmatrix} 011100 \\ 1 \end{smallmatrix} \right\rangle$ ,  $\mathcal{I} = \left\langle \begin{smallmatrix} 011100 & 111110 \\ 1 & 0 \end{smallmatrix} \right\rangle$ , and  $\mathcal{I} = \left\langle \begin{smallmatrix} 011100 & 111111 \\ 1 & 0 \end{smallmatrix} \right\rangle$ .

4.8. Next, we consider the case when  $\mathcal{I} \subseteq \mathcal{I}_5$ . In case of  $F_4$  and  $E_6$  one readily checks that there is no maximal standard parabolic subsystem  $\Phi_0$  such that Condition 1.10 is satisfied.

$\Phi$	$\beta \in \mathcal{I}$	$\Phi_0$	$\Phi_0^c \cap \mathcal{I}^c$
$F_4$	1111	$C_3$	1000, 1100, 1110, 1120, 1220
	1120	$C_3$	1000, 1100, 1110, 1111
	0121	$B_3$	0001, 0011, 0111, 1111
$E_6$	$\begin{smallmatrix} 11110 \\ 0 \end{smallmatrix}$	$D_5$	$\begin{smallmatrix} 10000 & 11000 & 11100 & 11100 \\ 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 11100 \\ 1 \end{smallmatrix}$	$D_5$	$\begin{smallmatrix} 10000 & 11000 & 11100 & 11110 & 11111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 01110 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 00111 \\ 1 \end{smallmatrix}$	$D'_5$	$\begin{smallmatrix} 00001 & 00011 & 00111 & 01111 & 11111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 01111 \\ 0 \end{smallmatrix}$	$D'_5$	$\begin{smallmatrix} 00001 & 00011 & 00111 & 00111 \\ 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$
$E_7$	$\begin{smallmatrix} 111100 \\ 0 \end{smallmatrix}$	$D_6$	$\begin{smallmatrix} 100000 & 110000 & 111000 & 111000 \\ 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 111000 \\ 1 \end{smallmatrix}$	$D_6$	$\begin{smallmatrix} 100000 & 110000 & 111000 & 111100 & 111110 & 111111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 011100 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 011110 \\ 0 \end{smallmatrix}$	$E_6$	$\begin{smallmatrix} 000001 & 000011 & 000111 & 001111 & 001111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 001110 \\ 1 \end{smallmatrix}$	$E_6$	$\begin{smallmatrix} 000001 & 000011 & 000111 & 001111 & 011111 & 111111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 001111 \\ 0 \end{smallmatrix}$	$E_6$	$\begin{smallmatrix} 000001 & 000011 & 000111 \\ 0 & , & 0 & , & 0 \end{smallmatrix}$
$E_8$	$\begin{smallmatrix} 1111000 \\ 0 \end{smallmatrix}$	$D_7$	$\begin{smallmatrix} 1000000 & 1100000 & 1110000 & 1110000 \\ 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 1110000 \\ 1 \end{smallmatrix}$	$D_7$	$\begin{smallmatrix} 1000000 & 1100000 & 1110000 & 1111000 & 1111100 & 1111110 & 1111111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 0111000 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 0011100 \\ 1 \end{smallmatrix}$	$E_7$	$\begin{smallmatrix} 0000001 & 0000011 & 0000111 & 0001111 & 0011111 & 0111111 & 1111111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 0111100 \\ 0 \end{smallmatrix}$	$E_7$	$\begin{smallmatrix} 0000001 & 0000011 & 0000111 & 0001111 & 0011111 & 0011111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 0011110 \\ 0 \end{smallmatrix}$	$E_7$	$\begin{smallmatrix} 0000001 & 0000011 & 0000111 & 0001111 \\ 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 0001111 \\ 0 \end{smallmatrix}$	$E_7$	$\begin{smallmatrix} 0000001 & 0000011 & 0000111 \\ 0 & , & 0 & , & 0 \end{smallmatrix}$

TABLE 7. Condition 1.10 for  $\mathcal{I} \subseteq \mathcal{I}_4$  and  $\mathcal{I} \not\subseteq \mathcal{I}_5$

It follows from Remark 3.10, Theorem 1.9, [BC12, Cor. 5.15], and the data in Tables 6 and 7, that for  $F_4$  and  $E_6$ , there are at most 20, respectively 62,  $\mathcal{A}_{\mathcal{I}}$  which might fail to be inductively free. This number is listed in the last row of Table 5 labelled by  $*$ .

4.9. We continue by considering the case when  $\mathcal{I} \subseteq \mathcal{I}_5$  and  $\mathcal{I} \not\subseteq \mathcal{I}_6$  for  $E_7$  and  $E_8$ . Then there is a root of height 5 in  $\mathcal{I}$  but no root of smaller height. We list all cases when Condition



1.10 is satisfied in Table 8 below. The notation is as in the previous tables. In particular, as before, “ $\times$ ” indicates that Condition 1.10 fails for any choice of maximal parabolic subsystem.

$\Phi$	$\beta \in \mathcal{I}$	$\Phi_0$	$\Phi_0^c \cap \mathcal{I}^c$
$E_7$	$\begin{smallmatrix} 111110 \\ 0 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 111100 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 011110 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 012100 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 011111 \\ 0 \end{smallmatrix}$	$E_6$	$\begin{smallmatrix} 000001 & 000011 & 000111 & 001111 & 001111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 001111 \\ 1 \end{smallmatrix}$	$E_6$	$\begin{smallmatrix} 000001 & 000011 & 000111 & 001111 & 011111 & 111111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}$
$E_8$	$\begin{smallmatrix} 1111100 \\ 0 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 1111000 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 0111100 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 0121000 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 0011110 \\ 1 \end{smallmatrix}$	$E_7$	$\begin{smallmatrix} 0000001 & 0000011 & 0000111 & 0001111 & 0011111 & 0111111 & 1111111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}$
	$\begin{smallmatrix} 0111110 \\ 0 \end{smallmatrix}$	$E_7$	$\begin{smallmatrix} 0000001 & 0000011 & 0000111 & 0001111 & 0011111 & 0011111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$
	$\begin{smallmatrix} 0011111 \\ 0 \end{smallmatrix}$	$E_7$	$\begin{smallmatrix} 0000001 & 0000011 & 0000111 & 0001111 \\ 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}$

TABLE 8. Condition 1.10 for  $\mathcal{I} \subseteq \mathcal{I}_5$  and  $\mathcal{I} \not\subseteq \mathcal{I}_6$

If  $\mathcal{I}$  is an ideal as above, containing more than one root  $\beta$  of height 5 so that already for the smaller ideal  $\langle \beta \rangle$  there is no parabolic subsystem  $\Phi_0$  which satisfies Condition 1.10, then it is readily checked that this is also the case for the larger ideal  $\mathcal{I}$ . On the other hand if  $\mathcal{I}$  is an ideal which contains roots  $\beta$  of height 5 so that Condition 1.10 holds for the ideal  $\langle \beta \rangle$  for some such  $\beta$  and for some choice of maximal parabolic subsystem  $\Phi_0$ , then Condition 1.10 also holds for the larger ideal  $\mathcal{I}$  and the same  $\Phi_0$ .

For instance, consider the ideal  $\mathcal{I} = \left\langle \begin{smallmatrix} 111110 \\ 0 \end{smallmatrix}, \begin{smallmatrix} 001111 \\ 1 \end{smallmatrix} \right\rangle$  (cf. the first and last rows of the cases for  $E_7$  in Table 8). We see that for  $\Phi_0$  of type  $E_6$  Condition 1.10 holds and  $\Phi_0^c \cap \mathcal{I}^c = \left\{ \begin{smallmatrix} 000001 & 000011 & 000111 & 001111 & 011111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix} \right\}$ .

For  $\Phi$  of type  $E_7$ , respectively of type  $E_8$ , there are 323, respectively 355, ideals of this kind for which Condition 1.10 does not hold for any choice of a maximal standard parabolic subsystem  $\Phi_0$  of  $\Phi$ .

4.10. Next, we consider the case when  $\mathcal{I} \subseteq \mathcal{I}_6$ . One checks that in case of  $E_7$  there is no maximal parabolic subsystem  $\Phi_0$  such that Condition 1.10 is satisfied. For  $E_8$  there is only one case of such an ideal  $\mathcal{I}$ . This is listed in Table 9 below. Note that for  $E_8$  one checks

that if  $\mathcal{I} \subseteq \mathcal{I}_7$ , then again there is no maximal standard parabolic subsystem  $\Phi_0$  such that Condition 1.10 is satisfied.

For  $\Phi$  of type  $E_7$ , respectively of type  $E_8$ , there are 403, respectively 1317, ideals of this kind for which Condition 1.10 does not hold for any choice of a maximal standard parabolic subsystem  $\Phi_0$  of  $\Phi$ .

$\Phi$	$\beta \in \mathcal{I}$	$\Phi_0$	$\Phi_0^c \cap \mathcal{I}^c$
$E_8$	$\begin{smallmatrix} 1111110 \\ 0 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 1111100 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 0111110 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 1121000 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 0121100 \\ 1 \end{smallmatrix}$	$\times$	$\times$
	$\begin{smallmatrix} 0111111 \\ 0 \end{smallmatrix}$	$E_7$	$\begin{smallmatrix} 0000001 & 0000011 & 0000111 & 0001111 & 0011111 & 0011111 \\ 0 & , & 0 & , & 0 & , & 0 & , & 0 & , & 1 \end{smallmatrix}$

TABLE 9. Condition 1.10 for  $\mathcal{I} \subseteq \mathcal{I}_6$  and  $\mathcal{I} \not\subseteq \mathcal{I}_7$

4.11. Finally, by induction on the rank and thanks to Theorem 1.9 and [BC12, Cor. 5.15] along with the results from §§4.4 – 4.10, including the data in Tables 6 – 9, it follows that for  $E_7$ , respectively  $E_8$ , there are at most  $403 + 323 + 3 - 2 = 727$ , respectively  $4500 + 1317 + 355 + 8 - 2 = 6178$  arrangements of ideal type  $\mathcal{A}_{\mathcal{I}}$  which might fail to be inductively free. This number of instances is listed in the last row of Table 5 labelled by  $*$ .

Finally, [BC12, Cor. 5.15] along with Theorem 1.9 complete the proof Theorem 1.14.

**Remark 4.1.** Above we have dealt with all instances when Condition 1.10 holds in the exceptional types. As observed this leads to a surprisingly small number of ideals left to be considered further. This number is shown in the last row of Table 5 above labeled with  $*$ . Here we have already taken the cases  $\mathcal{I}_{h-1} = \{\theta\}$  and  $\mathcal{I}_h = \emptyset$  into account as well which give inductively free arrangements of ideal type, by Theorem 1.9 and [BC12, Cor. 5.15].

For each of these remaining instances other than the outstanding 6178 for type  $E_8$ , T. Hoge was able to check that  $\mathcal{A}_{\mathcal{I}}$  is inductively free by computational means.

Finally Theorem 1.15 follows from Theorem 1.14 and Remark 4.1.

## 5. THE POINCARÉ POLYNOMIAL $\mathcal{I}(t)$ OF $\mathcal{I}$

**5.1. Rank-generating functions of posets of regions of real arrangements.** In this subsection let  $\mathcal{A}$  be a hyperplane arrangement in the real vector space  $V = \mathbb{R}^\ell$ . A *region* of  $\mathcal{A}$  is a connected component of the complement  $M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H$  of  $\mathcal{A}$ . Let  $\mathcal{R} := \mathcal{R}(\mathcal{A})$  be the set of regions of  $\mathcal{A}$ . For  $R, R' \in \mathcal{R}$ , we let  $\mathcal{S}(R, R')$  denote the set of hyperplanes in  $\mathcal{A}$  separating  $R$  and  $R'$ . Then with respect to a choice of a fixed base region  $B$  in  $\mathcal{R}$ , we can partially order  $\mathcal{R}$  as follows:

$$R \leq R' \quad \text{if} \quad \mathcal{S}(B, R) \subseteq \mathcal{S}(B, R').$$

Endowed with this partial order, we call  $\mathcal{R}$  the *poset of regions of  $\mathcal{A}$  (with respect to  $B$ )* and denote it by  $P(\mathcal{A}, B)$ . This is a ranked poset of finite rank, where  $\text{rk}(R) := |\mathcal{S}(B, R)|$ , for  $R$  a region of  $\mathcal{A}$ , [Ed84, Prop. 1.1]. The *rank-generating function* of  $P(\mathcal{A}, B)$  is defined to be the following polynomial in  $\mathbb{Z}_{\geq 0}[t]$

$$\zeta(P(\mathcal{A}, B), t) := \sum_{R \in \mathcal{R}} t^{\text{rk}(R)}.$$

This poset along with its rank-generating function was introduced by Edelman [Ed84].

If  $\mathcal{A}$  is the product of arrangements  $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$  then so is the rank-generating function of its poset of regions

$$(5.1) \quad \zeta(P(\mathcal{A}, B), t) = \zeta(P(\mathcal{A}_1, B_1), t) \cdot \zeta(P(\mathcal{A}_2, B_2), t),$$

where  $B = B_1 \times B_2$ , see [Sta12, §3.3].

The following theorem due to Jambu and Paris, [JP95, Prop. 3.4, Thm. 6.1], was first shown by Björner, Edelman and Ziegler for  $\mathcal{A}$  supersolvable in [BEZ90, Thm. 4.4].

**Theorem 5.2.** *If  $\mathcal{A}$  is inductively factored, then there is a suitable choice of a base region  $B$  so that  $\zeta(P(\mathcal{A}, B), t)$  satisfies the multiplicative formula*

$$(5.3) \quad \zeta(P(\mathcal{A}, B), t) = \prod_{i=1}^{\ell} (1 + t + \dots + t^{e_i}),$$

where  $\{e_1, \dots, e_{\ell}\} = \exp \mathcal{A}$  is the set of exponents of  $\mathcal{A}$ .

We recall the inductive construction from the proof of [BEZ90, Thm. 4.4]. Let  $\mathcal{A}$  be a real arrangement of rank  $r$ . Suppose that  $X \in L(\mathcal{A})$  is modular of rank  $r - 1$ . Let  $\mathcal{A}_0 := \mathcal{A}_X$ . Then there is an order-preserving, surjective map

$$\pi : P(\mathcal{A}, B) \rightarrow P(\mathcal{A}_0, \pi(B)),$$

which is induced by inclusion of regions. For  $R \in \mathcal{R}$ , let  $\mathcal{F}(R) := \pi^{-1}(\pi(R))$  be its fibre under  $\pi$ . We say that  $B$  in  $\mathcal{R}$  is *canonical (with respect to  $\mathcal{A}_0$ )* provided  $\mathcal{F}(B)$  is linearly ordered within  $P(\mathcal{A}, B)$ . The proof of the following lemma is just the argument from [BEZ90, Thm. 4.4].

**Lemma 5.4.** *Let  $\mathcal{A}$  be a real arrangement of rank  $r$ . Suppose that  $X \in L(\mathcal{A})$  is modular of rank  $r - 1$ . Let  $\mathcal{A}_0 := \mathcal{A}_X$  and set  $e_{\ell} := |\mathcal{A} \setminus \mathcal{A}_0|$ . Then there is a region  $B$  in  $\mathcal{R}$  which is canonical with respect to  $\mathcal{A}_0$ . In addition, setting  $B_0 := \pi(B)$ , we have the following:*

- (i)  $\zeta(P(\mathcal{A}, B), t) = \zeta(P(\mathcal{A}_0, B_0), t) \cdot (1 + t + \dots + t^{e_{\ell}})$ .
- (ii) *Suppose that there are non-negative integers  $e_1, \dots, e_{\ell-1}$  such that  $\zeta(P(\mathcal{A}_0, B_0), t)$  factors as in (5.3). Then  $\zeta(P(\mathcal{A}, B), t)$  also factors as in (5.3) (with  $e_{\ell} = |\mathcal{A} \setminus \mathcal{A}_0|$ ).*

*Proof.* Since  $X \in L(\mathcal{A})$  is modular of rank  $r - 1$ , any two hyperplanes in  $\mathcal{A} \setminus \mathcal{A}_0$  do not intersect in a region of  $\mathcal{R}(\mathcal{A}_0)$ , by Lemma 2.1. It follows that for any  $R \in \mathcal{R}$  the regions in  $\mathcal{F}(R)$  are linearly ordered by adjacency and each fibre has the same length  $e_{\ell} = |\mathcal{A} \setminus \mathcal{A}_0|$ . Fix a region  $B$  in  $\mathcal{F}(R)$  at the end of such a linear chain of adjacencies for some  $R \in \mathcal{R}$ . Then  $B$  is canonical with respect to  $\mathcal{A}_0$ . Moreover, for  $R \in \mathcal{R}$ , calculating the rank in  $P(\mathcal{A}, B)$  with respect to  $B$ , we have  $\text{rk}(R) = \text{rk}_0(\pi(R)) + \text{rk}_{\mathcal{F}(R)}(R)$ , where  $\text{rk}_0(\pi(R))$  is the rank of  $\pi(R)$  in  $P(\mathcal{A}_0, B_0)$  with respect to  $B_0$  and  $\text{rk}_{\mathcal{F}(R)}(R)$  is the rank of  $R$  in its fibre which is induced from the rank in  $P(\mathcal{A}, B)$ . Hence the rank-generating function  $\zeta(P(\mathcal{A}, B), t)$  of the

poset of regions  $P(\mathcal{A}, B)$  with respect to  $B$  is the product of the rank-generating function  $\zeta(P(\mathcal{A}_0, B_0), t)$  of the poset of regions  $P(\mathcal{A}_0, B_0)$  with respect to  $B_0$  and the rank-generating function for each fibre. Thus, since  $B$  is canonical with respect to  $\mathcal{A}_0$ , (i) follows.

Finally, (ii) follows directly from (i).  $\square$

The relevance of Lemma 5.4 stems from the fact that it can be used to show that the rank generating function  $\zeta(P(\mathcal{A}, B), t)$  factors as in (5.3) even when  $\mathcal{A}_0$  is not inductively factored (or supersolvable). We illustrate this with an example from the setting from §3 where we also use the notation from that section.

**Example 5.5.** Let  $n \geq 5$  and  $\Phi$  be of type  $D_n$  and let  $\Phi_0$  be the standard subsystem of type  $D_{n-1}$ . Let  $\mathcal{I}$  be the ideal in  $\Phi^+$  generated by  $e_1 - e_j$  for  $2 \leq j \leq n$ . Then Condition 1.10 holds. Note that with the notation from §3, we have  $\mathcal{I}_0 = \emptyset$ , so that  $\mathcal{A}_{\mathcal{I}_0} = \mathcal{A}(D_{n-1})$  is the reflection arrangement of  $\Phi_0$ . In particular,  $\mathcal{A}_{\mathcal{I}_0}$  is not supersolvable and not inductively factored. For  $B$  a region of  $\mathcal{A}_{\mathcal{I}}$  containing the fundamental dominant Weyl chamber of  $\Phi$  and for  $\mathcal{A}_0 := \mathcal{A}_{\mathcal{I}_0}$  and  $B_0 = \pi(B)$  as above, we see from (5.7) that  $\zeta(P(\mathcal{A}_0, B_0), t)$  is just the Poincaré polynomial of  $W(D_{n-1})$  which factors according to (1.20). Therefore, by Lemma 5.4(ii),  $\zeta(P(\mathcal{A}_{\mathcal{I}}, B), t)$  factors as in (5.3) as well. As a consequence of (5.7), Conjecture 1.22 holds in these instances.

**5.2. The Poincaré polynomial  $\mathcal{I}(t)$  and the rank-generating function of the poset of regions of  $\mathcal{A}_{\mathcal{I}}$ .** There is a connection between the combinatorics of an arrangement of ideal type  $\mathcal{A}_{\mathcal{I}}$ , by way of the Poincaré polynomial  $\mathcal{I}(t)$  of  $\mathcal{I}$  and the geometry of the arrangement  $\mathcal{A}_{\mathcal{I}}$ , via the theory of rank-generating functions of the poset of regions of  $\mathcal{A}_{\mathcal{I}}$ , as follows.

**Remarks 5.6.** (i). In [ST06, §12], Sommers and Tymoczko observed that if we fix a region  $B$  of the set of regions  $\mathcal{R} = \mathcal{R}(\mathcal{A}_{\mathcal{I}})$  of  $\mathcal{A}_{\mathcal{I}}$  which contains the dominant Weyl chamber of  $\Phi$ , then thanks to [ST06, Prop. 6.1], the rank-generating function of the poset of regions  $P(\mathcal{A}_{\mathcal{I}}, B)$  of  $\mathcal{A}_{\mathcal{I}}$  with respect to  $B$  is just the Poincaré polynomial  $\mathcal{I}(t)$  of  $\mathcal{I}$  introduced in (1.21):

$$(5.7) \quad \zeta(P(\mathcal{A}_{\mathcal{I}}, B), t) = \sum_{R \in \mathcal{R}} t^{\text{rk}(R)} = \sum_{S \in \mathcal{W}^{\mathcal{I}}} t^{|S|} = \mathcal{I}(t).$$

(ii). Thanks to Theorem 5.2 and (5.7), the Poincaré polynomial  $\mathcal{I}(t)$  of a supersolvable or inductively factored arrangement of ideal type  $\mathcal{A}_{\mathcal{I}}$  satisfies (1.23) and thus for all supersolvable or inductively factored  $\mathcal{A}_{\mathcal{I}}$  Conjecture 1.22 holds. Thus Theorem 1.31 gives further evidence for this conjecture.

(iii). Theorem 1.24 follows from Theorems 1.5 and 5.2 combined with (5.7).

In view of Remark 5.6(i), Theorem 1.26 is an easy consequence of Lemmas 3.4 and 5.4.

*Proof of Theorem 1.26.* Thanks to Remark 5.6(i), it suffices to consider the rank-generating function  $\zeta(P(\mathcal{A}_{\mathcal{I}}, B), t)$  of the poset of regions of  $\mathcal{A}_{\mathcal{I}}$  for a region  $B$  of  $\mathcal{R}(\mathcal{A}_{\mathcal{I}})$  containing the dominant Weyl chamber of  $\Phi$ . If  $\mathcal{A}_{\mathcal{I}}$  is reducible, then  $\mathcal{A}_{\mathcal{I}}$  is the product of two arrangements of ideal type of smaller rank root systems. The multiplicative property for  $\zeta(P(\mathcal{A}_{\mathcal{I}}, B), t)$  then follows from (5.1) and the inductive hypothesis made in the statement.

So suppose that  $\mathcal{A}_{\mathcal{I}}$  is irreducible and that Condition 1.10 holds. It then follows from the inductive hypothesis and Lemmas 3.4 and 5.4(ii) that  $\zeta(P(\mathcal{A}_{\mathcal{I}}, B), t)$  factors as in (5.3), as claimed.  $\square$

Theorem 1.27 follows equally readily.

*Proof of Theorem 1.27.* In cases (i) and (ii), we argue as in the proof of Theorem 1.26.

If  $\mathcal{I} = \{\theta\}$  or  $\mathcal{I} = \emptyset$ , then the result follows from Theorem 1.25 and Solomon's formula (1.20), respectively.

Because the very same instances have already been accounted for in Table 1 in connection with our analysis of the inductively free  $\mathcal{A}_{\mathcal{I}}$  for  $\Phi$  of exceptional type in Theorem 1.14, the final statement follows from the determination of all the instances covered in (i) and (ii) from Section 4.  $\square$

**Remarks 5.8.** (i). The relevance in studying the class of inductively factored real arrangements for our purpose lies in the fact that it is the largest class of free arrangements which satisfies the multiplicative formula (5.3). This formula does not hold in general for the larger class of inductively free arrangements. For instance, the real simplicial arrangement “ $A_4(17)$ ” from Grünbaum's list [Gr71] does not factor as in (5.3), as observed by Terao, see [BEZ90, p. 277]. T. Hoge has checked that this arrangement is inductively free.

Therefore, in the context of arrangements of ideal type, where we can't expect more than inductive freeness in general, according to our discussion above, the general factorization property from Theorem 1.26 (predicted for all  $\mathcal{A}_{\mathcal{I}}$  by Conjecture 1.22) shows that the arrangements of ideal type are rather special in this regard.

(ii). The multiplicative formula (5.3) for a free real arrangement  $\mathcal{A}$  involving the exponents frequently plays a crucial role in related contexts.

For instance, for a fixed  $w$  in  $W$ , define its Poincaré polynomial by  $P_w(t) = \sum_{x \leq w} t^{\ell(x)}$ , where  $\leq$  is the Bruhat-Chevalley order on  $W$ . E.g., for  $w_0$  the longest word in  $W$ ,  $P_{w_0}(t) = W(t)$ , see (1.19). In geometric terms,  $P_w(t^2)$  is the Poincaré polynomial of the Schubert variety  $X(w) = \overline{BwB}/B$  in the full flag manifold  $G/B$  of the semisimple Lie group  $G$  associated with  $w$  in the Weyl group  $W$  of  $G$ .

Consider the subarrangement  $\mathcal{A}_w$  of the Weyl arrangement  $\mathcal{A}(W)$  given by the hyperplanes corresponding to the set of roots  $N(w)$  introduced in (1.18). According to a criterion due to Carrell and Peterson [Car94], the Schubert variety  $X(w)$  is rationally smooth if and only if  $P_w(t)$  is palindromic.

All instances are known when  $P_w(t)$  admits a factorization analogous to (1.20). This is the case precisely when  $X(w)$  is rationally smooth. Equivalently, this is the case precisely when  $\mathcal{A}_w$  is free and the exponents then make an appearance in this analogue of (1.20) for  $P_w(t)$ , cf. [Slo15, Thm. 3.3]. This is also the case precisely when the rank-generating function of the poset of regions of  $\mathcal{A}_w$  coincides with  $P_w(t)$ , [OY10].

## 6. SUPERSOLVABLE AND INDUCTIVELY FACTORED $\mathcal{A}_{\mathcal{I}}$

In this final section we prove Theorems 1.30 and 1.31. First we consider the situation for  $\Phi$  of type  $D_4$ .

**Lemma 6.1.** *Let  $\Phi$  be of type  $D_4$ . Then  $h = 6$ . Then all arrangements of ideal type  $\mathcal{A}_{\mathcal{I}}$  but the three corresponding to  $\mathcal{I}_6 = \emptyset$ ,  $\mathcal{I}_5$ , and  $\mathcal{I}_4$  are supersolvable and all but the ones corresponding to  $\mathcal{I}_6 = \emptyset$  and  $\mathcal{I}_5$ , are inductively factored. Thus of the 50 arrangements of ideal type  $\mathcal{A}_{\mathcal{I}}$  in type  $D_4$ , 47 are supersolvable and 48 are inductively factored.*

*Proof.* According to Table 4, there are 50 arrangements of ideal type in this case and 20 strictly positive ideals. Since each proper parabolic subsystem of  $D_4$  is a product of type  $A$

subsystems, it follows from Theorem 1.5 and Proposition 2.14 (see Example 3.7) that the 30 arrangements  $\mathcal{A}_{\mathcal{I}}$ , where  $\mathcal{I}$  is not strictly positive are all supersolvable.

Next, consider all  $\mathcal{I}$  with  $\mathcal{I} \subseteq \mathcal{I}_2$  and  $\mathcal{I} \not\subseteq \mathcal{I}_3$ . Then there is a root of height 2 in  $\mathcal{I}$ . Again, since  $\mathcal{A}_{\mathcal{I}}$  is a product of smaller rank arrangements of ideal type, it follows from Theorem 1.5 and Proposition 2.14 that all  $\mathcal{A}_{\mathcal{I}}$  of this kind are supersolvable.

Next suppose that  $\mathcal{I} \subseteq \mathcal{I}_3$  and  $\mathcal{I} \not\subseteq \mathcal{I}_4$ . Then there is a root of height 3 in  $\mathcal{I}$ . There are 7 such ideals. One checks that there is always a rank 3 subsystem  $\Phi_0$  of type  $A_3$ , so that Condition 1.10 is satisfied. Thus, by Theorems 1.5 and 1.12(i), each such arrangement  $\mathcal{A}_{\mathcal{I}}$  is supersolvable.

Summarizing the observations above,  $\mathcal{A}_{\mathcal{I}}$  is supersolvable for any  $\mathcal{I} \not\subseteq \mathcal{I}_4$ , thanks to Theorems 1.5 and 1.12(i) and Proposition 2.14. This applies to 47 out of the 50 ideals. The 3 ideals left to consider are  $\mathcal{I}_4 = \{e_1 + e_3, e_1 + e_2\}$ ,  $\mathcal{I}_5 = \{e_1 + e_2\}$ , and  $\mathcal{I}_6 = \emptyset$ . Of course,  $\mathcal{A}_{\mathcal{I}_6} = \mathcal{A}(D_4)$  is neither supersolvable, nor inductively factored, e.g. see [JT84, Ex. 5.5] or Theorem 2.28. One checks directly that  $\mathcal{A}_{\mathcal{I}_4}$  is not supersolvable. In addition, one can show that  $\mathcal{A}_{\mathcal{I}_5}$  is not nice, so in particular, is not inductively factored, nor supersolvable. In contrast, for  $\mathcal{I}_4$  one can show that

$$(6.2) \quad \{H_{e_2-e_3}\}, \{H_{e_3-e_4}, H_{e_2-e_4}, H_{e_2+e_3}\}, \{H_{e_1-e_2}, H_{e_1-e_3}, H_{e_1-e_4}\}, \{H_{e_3+e_4}, H_{e_2+e_4}, H_{e_1+e_4}\}$$

is an inductive factorization of  $\mathcal{A}_{\mathcal{I}_4}$ , where the notation is as in [Bou68, Planche IV]. We show this directly as follows. Let  $\delta = e_2 + e_3$ . By the argument above, for  $\mathcal{I} = \langle \delta \rangle$  we have  $\mathcal{A}_{\mathcal{I}}$  is supersolvable with exponents  $\exp \mathcal{A}_{\mathcal{I}} = \{1, 2, 3, 3\}$ . Note that  $\mathcal{I}_4 \setminus \{\delta\} = \mathcal{I}^c$ . Consider the triple of  $\mathcal{A}_{\mathcal{I}_4}$  with respect to  $H_{\delta}$ . Then we have  $(\mathcal{A}_{\mathcal{I}_4}, \mathcal{A}'_{\mathcal{I}_4} = \mathcal{A}_{\mathcal{I}}, \mathcal{A}''_{\mathcal{I}_4})$ . Since  $\mathcal{A}'_{\mathcal{I}_4} = \mathcal{A}_{\mathcal{I}}$  is supersolvable, it admits a canonical inductive factorization given by

$$\{H_{e_2-e_3}\}, \{H_{e_3-e_4}, H_{e_2-e_4}\}, \{H_{e_1-e_2}, H_{e_1-e_3}, H_{e_1-e_4}\}, \{H_{e_3+e_4}, H_{e_2+e_4}, H_{e_1+e_4}\},$$

stemming from a maximal chain of modular elements in  $L(\mathcal{A}_{\mathcal{I}})$ , see Proposition 2.20, see also [Ter92, Ex. 2.4] and [HR16a, Prop. 3.11]. Moreover, one checks that  $\mathcal{A}''_{\mathcal{I}_4}$  is inductively factored with inductive factorization given by

$$\{\bar{H}_{e_2-e_3}\}, \{\bar{H}_{e_1-e_2}, \bar{H}_{e_1-e_3}, \bar{H}_{e_1-e_4}\}, \{\bar{H}_{e_3+e_4}, \bar{H}_{e_2+e_4}, \bar{H}_{e_1+e_4}\}.$$

It follows from Theorem 2.18 that the partition of  $\mathcal{A}_{\mathcal{I}}$  given in (6.2) defines an inductive factorization of  $\mathcal{A}_{\mathcal{I}_4}$ .

Thus precisely 47 of the 50 arrangements  $\mathcal{A}_{\mathcal{I}}$  are supersolvable and precisely 48 are inductively factored.  $\square$

**Corollary 6.3.** *Conjecture 1.22 holds for  $\Phi$  of type  $D_4$ .*

*Proof.* Let  $\Phi$  be of type  $D_4$  and let  $\mathcal{I}$  be an ideal in  $\Phi^+$ . It follows from Lemma 6.1 that either  $\mathcal{A}_{\mathcal{I}}$  is inductively factored, or else  $\mathcal{I} = \{\theta\}$ , or  $\mathcal{I} = \emptyset$ . So the result follows from Theorem 5.2 and (5.7), Theorem 1.25, and (1.20), respectively.  $\square$

**Remark 6.4.** We note that one can show that out of all  $\binom{2n-1}{n} + \binom{2n-2}{n}$  ideals in type  $D_n$ , there are just  $2^{n-2}$  ideals for which Condition 1.10 fails with respect to  $\Phi_0$  being the standard subsystem of type  $D_{n-1}$ . However, if  $\mathcal{I}$  is the largest such ideal, i.e.  $\mathcal{I} = \langle e_{n-2} + e_{n-1} \rangle$ , then  $\mathcal{A}_{\mathcal{I}}$  is supersolvable, see Example 3.9. Thus Conjecture 1.22 holds in this case, by Theorem 5.2 and (5.7). Likewise, for  $\mathcal{I} = \{\theta\}$ , or  $\mathcal{I} = \emptyset$ , Conjecture 1.22 holds, by Theorem 1.25 and (1.20).



Thus, assuming the factorization result from Conjecture 1.22 for  $D_{n-1}$ , it follows from Theorem 1.26 that it holds for all instances in  $D_n$  as well with at most  $2^{n-2} - 3$  exceptions.

A. Schauenburg was able to check that Conjecture 1.22 always holds also in these additional  $2^{n-2} - 3$  cases for  $5 \leq n \leq 7$  by direct computational means.

We are now in a position to prove Theorems 1.30 and 1.31.

*Proof of Theorem 1.30.* We prove both parts simultaneously. For  $\Phi$  of type  $D_4$ , the result follows from Lemma 6.1. Now let  $\Phi$  be as in the statement of rank at least 5 and let  $\Phi_0$  be the standard subsystem of  $\Phi$  of type  $D_4$ . Since,  $\mathcal{I}_0 = \mathcal{I} \cap \Phi_0^+$  also consists of roots of height at least 4, respectively 5, by the result for  $D_4$ ,  $\mathcal{A}_{\mathcal{I}_0}$  is not supersolvable, respectively not inductively factored. It now follows from Corollary 3.2(i) and (iii) that the same holds for  $\mathcal{A}_{\mathcal{I}}$ .  $\square$

**Remark 6.5.** We can strengthen Theorem 1.30(ii) slightly as follows. Thanks to Lemma 6.1,  $\mathcal{A}_{\mathcal{I}}$  is not nice for  $\mathcal{I} \subseteq \mathcal{I}_5$ . This also holds for  $\Phi$  as in the statement of Theorem 1.30, arguing as in the proof above and using Remark 2.17 and Lemma 3.1.

*Proof of Theorem 1.31.* (i). We argue by induction on the rank. For  $\Phi$  of type  $D_4$ , the result follows from Lemma 6.1. Now let  $\Phi$  be of type  $D_n$  for  $n \geq 5$ ,  $E_6$ ,  $E_7$ , or  $E_8$  and suppose that the result is true for smaller rank root systems of these kinds.

If  $\mathcal{I}$  contains a root of height 1 or 2, then it follows that  $\mathcal{A}_{\mathcal{I}}$  is the product of arrangements of ideal type for smaller rank root systems of types  $A$ ,  $D$  or  $E$  and each ideal corresponding to a factor still contains all roots of height 3 of the smaller rank root system. Thus by induction on the rank, Theorem 1.5, and Proposition 2.14, the product  $\mathcal{A}_{\mathcal{I}}$  is also supersolvable.

It remains to consider the case when  $\mathcal{I} = \mathcal{I}_3$ . So assume  $\mathcal{I} = \mathcal{I}_3$  and let  $\Phi_0$  be the standard subsystem of  $\Phi$  of type  $D_{n-1}$ ,  $D_5$ ,  $D_6$ , or  $D_7$ , respectively. Then Condition 1.10 is satisfied. Therefore, since  $\mathcal{A}_{\mathcal{I}_0}$  is supersolvable by induction (for  $\mathcal{I}_0$  is the ideal  $\Phi_0^+ \cap \mathcal{I}_3$  in  $\Phi_0$ ), so is  $\mathcal{A}_{\mathcal{I}}$ , thanks to Theorem 1.12(i).

(ii). Again, we argue by induction on the rank. For  $\Phi$  of type  $D_4$ , the result follows again from Lemma 6.1. As above, if  $\mathcal{I}$  also contains a root of height 1 or 2, then  $\mathcal{A}_{\mathcal{I}}$  is the product of arrangements of ideal type for smaller rank root systems of types  $A$ ,  $D$  or  $E$  and each ideal corresponding to a factor still contains all roots of height 4 of the smaller rank root system. Thus by induction on the rank, Theorem 1.5, and Proposition 2.26, the product  $\mathcal{A}_{\mathcal{I}}$  is also inductively factored.

Now suppose that  $\mathcal{I} \subseteq \mathcal{I}_3$  but  $\mathcal{I} \not\subseteq \mathcal{I}_4$ , so that  $\mathcal{I}$  still contains a root of height 3 but none of smaller height.

Suppose first that  $\Phi$  is of type  $D_n$  for  $n \geq 5$ . If  $\mathcal{I}$  contains  $e_i - e_{i+3}$  for some  $1 \leq i \leq n-3$  or  $e_{n-3} + e_n$ , then for  $\Phi_0$  the standard subsystem of type  $D_{n-1}$ , Condition 1.10 holds. Therefore, since  $\mathcal{I}_0$  still contains all roots of height 4 in  $\Phi_0^+$ , it follows from induction on  $n$  that  $\mathcal{A}_{\mathcal{I}_0}$  is inductively factored. Thus so is  $\mathcal{A}_{\mathcal{I}}$ , thanks to Theorem 1.12(iii). Finally, if the only root of height 3 in  $\mathcal{I}$  is  $e_{n-2} + e_{n-1}$ , then let  $\Phi_0$  be one of the standard subsystems of type  $A_{n-1}$  in  $\Phi$ . Again it is easy to see that then Condition 1.10 is satisfied. Thus, thanks to Theorem 1.5,  $\mathcal{A}_{\mathcal{I}_0}$  is supersolvable and so is  $\mathcal{A}_{\mathcal{I}}$ , according to Theorem 1.12(i), and so  $\mathcal{A}_{\mathcal{I}}$  is inductively factored, by Proposition 2.20.

Now suppose  $\Phi$  is of type  $E$ . Since  $\mathcal{I}$  admits a root of height 3, it follows from Table 6 that there always exists a maximal parabolic subsystem  $\Phi_0$  so that Condition 1.10 holds. Since  $\Phi_0$  is of type  $A$ ,  $D$  or  $E$ , and  $\mathcal{I}_0$  still contains all the roots of height 4 of  $\Phi_0^+$ , so that by

induction on the rank, Theorem 1.5 and the case for type  $D_n$  just proved,  $\mathcal{A}_{\mathcal{I}_0}$  is inductively factored, then so is  $\mathcal{A}_{\mathcal{I}}$ , according to Theorem 1.12(iii).

It remains to consider the case when  $\mathcal{I} = \mathcal{I}_4$ . So let  $\mathcal{I} = \mathcal{I}_4$ . Suppose first that  $\Phi$  is of type  $D_n$  for  $n \geq 5$ . Since  $\mathcal{I}$  contains  $e_1 - e_5$ , Condition 1.10 is valid for  $\Phi_0$  the standard subsystem of type  $D_{n-1}$ . Since  $\mathcal{I}_0$  is the ideal in  $\Phi_0^+$  consisting of all roots of height at least 4, it follows from induction on  $n$  that  $\mathcal{A}_{\mathcal{I}_0}$  is inductively factored, and thus so is  $\mathcal{A}_{\mathcal{I}}$ , thanks to Theorem 1.12(iii).

Finally, for  $\Phi$  of type  $E$  and  $\mathcal{I} = \mathcal{I}_4$ , it follows from Table 7 there always exists a maximal parabolic subsystem  $\Phi_0$  so that Condition 1.10 holds. Since  $\Phi_0$  is of type  $A$ ,  $D$  or  $E$ , and  $\mathcal{I}_0$  is the ideal in  $\Phi_0^+$  consisting of all roots of height at least 4, it follows from induction on the rank, Theorem 1.5 and the case for type  $D_n$  just proved that  $\mathcal{A}_{\mathcal{I}_0}$  is inductively factored, and once again, so is  $\mathcal{A}_{\mathcal{I}}$ , owing to Theorem 1.12(iii).  $\square$

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Thanks are also due to A. Schauenburg for computing the data in Table 5 for the cases when  $\mathcal{I} \subseteq \mathcal{I}_t$  for  $t \geq 3$  and for checking that Conjecture 1.22 holds for  $D_n$  for  $5 \leq n \leq 7$  and for  $E_7$ .

## REFERENCES

- [Abe16] T. Abe, *Divisionally free arrangements of hyperplanes*, Inventiones Math. **204**(1), (2016), 317–346.
- [ABC<sup>+</sup>16] T. Abe, M. Barakat, M. Cuntz, T. Hoge, and H. Terao, *The freeness of ideal subarrangements of Weyl arrangements*, JEMS, **18** (2016), no. 6, 1339–1348.
- [AT16] T. Abe and H. Terao, *Free filtrations of affine Weyl arrangements and the ideal-Shi arrangements*, J. Algebraic Combin. **43** (2016), no. 1, 33–44.
- [Aig79] M. Aigner, *Combinatorial Theory*, Grundlehren der math. Wissenschaften vol. 234 (1979).
- [BC12] M. Barakat and M. Cuntz, *Coxeter and crystallographic arrangements are inductively free*, Adv. Math **229** (2012), 691–709.
- [BEZ90] A. Björner, P. Edelman, and G. Ziegler, *Hyperplane arrangements with a lattice of regions*. Discrete Comput. Geom. **5** (1990), no. 3, 263–288.
- [Bou68] N. Bourbaki, *Éléments de mathématique. Groupes et algèbres de Lie. Chapitre IV-VI*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.
- [Car94] J. B. Carrell, *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, Algebraic groups and their generalizations: Classical methods (University Park, 1991), Proc. Sympos. Pure Math., **56**, part 1, pp. 5361, Amer. Math. Soc., Providence, RI, 1994.
- [CP00] P. Cellini and P. Papi, *ad-nilpotent ideals of a Borel subalgebra*. J. Algebra **225** (2000), no. 1, 130–141.
- [CP02] ———, *ad-nilpotent ideals of a Borel subalgebra. II*. J. Algebra **258** (2002), 112–121.
- [Ed84] P.H. Edelman, *A partial order on the regions of  $\mathbb{R}^n$  dissected by hyperplanes*. Trans. Amer. Math. Soc. **283** (1984), no. 2, 617–631.
- [ER94] P.H. Edelman and V. Reiner, *Free hyperplane arrangements between  $A_{n-1}$  and  $B_n$*  Math. Z. **215** (1994), no. 3, 347–365.

- [Gr71] B. Grünbaum, *Arrangements of Hyperplanes*, Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1971), pp. 41–106.
- [HR14] T. Hoge and G. Röhrle, *On supersolvable reflection arrangements*, Proc. AMS, **142** (2014), no. 11, 3787–3799.
- [HR15] ———, *On inductively free reflection arrangements*, J. Reine u. Angew. Math. **701** (2015), 205–220.
- [HR16a] ———, *Addition-Deletion Theorems for Factorizations of Orlik-Solomon Algebras and nice Arrangements*, European J. Combin. **55** (2016), 20–40.
- [HR16b] ———, *Nice Reflection Arrangements*, Electronic J. Combin. **23(2)**, (2016), # P2.9.
- [HRS17] T. Hoge, G. Röhrle and A. Schauenburg, *Inductive and Recursive Freeness of Localizations of multiarrangements*, to appear.
- [Hul16] A. Hultman, *Supersolvability and the Koszul property of root ideal arrangements*, Proc. AMS, **144** (2016), 1401–1413.
- [JP95] M. Jambu and L. Paris, *Combinatorics of Inductively Factored Arrangements*, European J. Combin. **16** (1995), 267–292.
- [JT84] M. Jambu and H. Terao, *Free arrangements of hyperplanes and supersolvable lattices*, Adv. in Math. **52** (1984), no. 3, 248–258.
- [Ko59] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*. Amer. J. Math. **81** (1959) 973–1032.
- [Ko61] ———, *Lie algebra cohomology and the generalized Borel-Weil theorem*. Ann. of Math. (2) **74** (1961) 329–387.
- [Mac72] I. G. Macdonald, *The Poincaré series of a Coxeter group*. Math. Ann. **199** (1972), 161–174.
- [MR17] T. Möller and G. Röhrle, *Localizations of inductively factored arrangements*, to appear.
- [OY10] S. Oh and H. Yoo, *Bruhat order, rationally smooth Schubert varieties, and hyperplane arrangements*. 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010), 965–972, Discrete Math. Theor. Comput. Sci. Proc., AN, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2010.
- [OST84] P. Orlik, L. Solomon, and H. Terao, *Arrangements of hyperplanes and differential forms*. Combinatorics and algebra (Boulder, Colo., 1983), 29–65, Contemp. Math., **34**, Amer. Math. Soc., Providence, RI, 1984.
- [OT92] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Springer-Verlag, 1992.
- [Shi97] J.-Y. Shi, *The number of  $\oplus$ -sign types*, Quart. J. Math. Oxford Ser. (2) **48** (1997), no. 189, 93–105.
- [Slo15] W. Slofstra, *Rationally smooth Schubert varieties and inversion hyperplane arrangements*, Adv. Math. **285** (2015), 709–736.
- [Sol66] L. Solomon, *The orders of the finite Chevalley groups*. J. Algebra **3** (1966) 376–393.
- [Som05] E. Sommers, *B-stable ideals in the nilradical of a Borel subalgebra*. Canad. Math. Bull. **48** (2005), no. 3, 460–472.
- [ST06] E. Sommers and J. Tymoczko, *Exponents for B-stable ideals*. Trans. Amer. Math. Soc. **358** (2006), no. 8, 3493–3509.
- [Sta72] R. P. Stanley, *Supersolvable lattices*, Algebra Universalis **2** (1972), 197–217.
- [Sta12] ———, *Enumerative combinatorics*. Volume 1. Second edition. Cambridge Studies in Advanced Mathematics, **49**. Cambridge University Press, Cambridge, 2012.
- [Sta99] ———, *Enumerative combinatorics*. Volume 2. Second edition. Cambridge Studies in Advanced Mathematics, **62**. Cambridge University Press, Cambridge, 1999.
- [Ste59] R. Steinberg, *Finite reflection groups*. Trans. Amer. Math. Soc., **91** (1959), 493–504.
- [Ste60] ———, *Invariants of finite reflection groups*, Canad. J. Math. **12**, (1960), 616–618.
- [Ter80] H. Terao, *Arrangements of hyperplanes and their freeness I*, J. Fac. Sci. Univ. Tokyo **27** (1980), 293–320.
- [Ter81] ———, *Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula*, Invent. Math. **63** no. 1 (1981) 159–179.
- [Ter92] ———, *Factorizations of the Orlik-Solomon Algebras*, Adv. in Math. **92**, (1992), 45–53.
- [Tym07] J. Tymoczko, *Paving Hessenberg varieties by affines*, Sel. math., New ser. **13**, (2007), 353–367.

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